

A semi-classical trace formula at a non-degenerate critical level.

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Abstract

We study the semi-classical trace formula at a critical energy level for a h -pseudo-differential operator whose principal symbol has a unique non-degenerate critical point for that energy. This leads to the study of Hamiltonian systems near equilibrium and near the non-zero periods of the linearized flow. The contributions of these periods to the trace formula are expressed in terms of degenerate oscillatory integrals. The new results obtained are formulated in terms of the geometry of the energy surface and the classical dynamics on this surface.

Key words: semi-classical analysis, trace formula, dynamical systems, degenerate oscillatory integrals.

1 Introduction

Let be P_h a h -pseudodifferential, or more generally h -admissible (see [15]), self-adjoint operator on \mathbb{R}^n . The semi-classical trace formula studies the asymptotic behavior, as h tends to 0, of the sums

$$\gamma(E, h, \varphi) = \sum_{|\lambda_j(h) - E| \leq \varepsilon} \varphi\left(\frac{\lambda_j(h) - E}{h}\right), \quad (1)$$

where the $\lambda_j(h)$ are the eigenvalues of P_h . Here we suppose that the spectrum is discrete in $[E - \varepsilon, E + \varepsilon]$, some sufficient conditions for this will be given below.

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Let be p the principal symbol of P_h and Φ_t the Hamilton flow of p . The semi-classical trace formula establishes a link between the asymptotic behavior of (1), as $h \rightarrow 0$, and the closed trajectories of Φ_t of energy E . An energy E is said to be regular when $\nabla p(x, \xi) \neq 0$ on Σ_E , where $\Sigma_E = \{(x, \xi) / p(x, \xi) = E\}$ is the surface of energy level E , and critical if it is not regular. The case of a regular energy has been intensively studied and explicit expressions in term of Φ_t are known for the leading term of (1), under suitable conditions on the flow and when the Fourier transform $\hat{\varphi}$ of φ is supported near a period, see, e.g., Gutzwiller [7], Balian and Bloch [1] for the physical literature, and from a mathematical point of view Brummelhuis and Uribe [3], Petkov and Popov [14], Charbonnel and Popov [4], Paul and Uribe [13].

Here we are interested in the case of a critical energy E_c of p . Brummelhuis, Paul and Uribe in [2] have studied the semi-classical trace formula at a critical energy for quite general operators but limited to "small times", that is for $\text{supp}(\hat{\varphi})$ contained in such a small neighborhood of the origin that the only period of the linearized flow in $\text{supp}(\hat{\varphi})$ is 0. Khuat-Duy, in [10], [11], has obtained the contributions of the non-zero periods of the linearized flow for arbitrary φ with compactly supported $\hat{\varphi}$, for Schrödinger operators $-\Delta + V(x)$ with $V(x)$ a non-degenerate potential. In this case the main contribution of such a period was obtained as a regularization of the Duistermaat-Guillemin density $\rho_t(x, \xi) = |\det(d\Phi_t(x, \xi) - \text{Id})|^{-\frac{1}{2}}$. Generalizing Khuat-Duy's result to more general operators was an open problem and is the purpose of this article. For a critical energy level E_c of an arbitrary h -admissible operator, and (x_0, ξ_0) a critical point of p , the closed trajectories of the linearized flow

$$\phi_t(u) = d_{x,\xi}\Phi_t(x_0, \xi_0)u = u, \quad u \in T_{(x_0, \xi_0)}(T^*\mathbb{R}^n), \quad t \neq 0, \quad (2)$$

are not necessarily generated by a positive quadratic form, contrary to the case of a Schrödinger operator, and will give rise to new contributions to the semi-classical trace formula, other than those obtained in [2], [10] and [11]. More precisely, viewing $d_{x,\xi}\Phi_t(x_0, \xi_0)$ as the Hamiltonian flow of the Hessian $d^2p(x_0, \xi_0)$, here interpreted as an intrinsic quadratic form, these new contributions to the trace formula arise from the non-trivial closed trajectories of $d_{x,\xi}\Phi_t(x_0, \xi_0)$ of zero energy

$$\begin{cases} \phi_T(u) = u, & T \neq 0, \\ d^2p(x_0, \xi_0)(u) = 0, & u \neq 0. \end{cases} \quad (3)$$

The reader can easily verify that the set of such u is empty in the case of a Schrödinger operator : this explains why the new contributions obtained here does not appear for these operators. We will show that the new contributions are supported in (E_c, x_0, ξ_0, T, u) with (T, u) satisfying (3) and can be expressed in term of $d^2p(x_0, \xi_0)$ and higher order derivatives of the flow in (x_0, ξ_0) .

2 General hypotheses and main results

Let P_h be in the class of h -admissible operators on \mathbb{R}^n with a real symbol. We refer to [15] for the principal notions of semi-classical analysis which we will use. We note p the principal symbol of P_h , and p^1 the sub-principal symbol. For E_c a critical energy of p we will study the asymptotic behavior of the spectral function $\gamma(E_c, h)$, defined by

$$\gamma(E_c, h) = \sum_{\lambda_j(h) \in [E_c - \varepsilon, E_c + \varepsilon]} \varphi\left(\frac{\lambda_j(h) - E_c}{h}\right), \quad (4)$$

under the following hypotheses which are classical in this context :

(H_1) $\exists \varepsilon_0 > 0$ such that $p^{-1}([E_c - \varepsilon_0, E_c + \varepsilon_0])$ is compact.

(H_2) $z_0 = (x_0, \xi_0)$ is the unique critical point of p on the energy surface Σ_{E_c} .

(H_3) z_0 is a non degenerate critical point of p and its Hessian $d^2p(z_0)$ is diagonal in some suitable set of local symplectic coordinates near z_0

$$p(x, \xi) = E_c + \frac{1}{2} \sum_{j=1}^n w_j ((x_j - x_{0,j})^2 + \sigma_j (\xi_j - \xi_{0,j})^2) + \mathcal{O}(\|(x - x_0, \xi - \xi_0)\|^3), \quad (5)$$

with $\sigma_j = \pm 1$ and $w_j \in \mathbb{R} \setminus \{0\}$.

(H_4) φ is in the Schwartz space $\mathcal{S}(\mathbb{R})$ with Fourier transform $\hat{\varphi} \in C_0^\infty(\mathbb{R})$.

By a classical result, see e.g. [15], the hypothesis (H_1) insures that the spectrum of P_h is discrete in $I_\varepsilon = [E_c - \varepsilon, E_c + \varepsilon]$ for $\varepsilon < \varepsilon_0$ and h small enough : this will be assumed in the following. We note $\text{Exp}(tH_f)$, with $H_f = \partial_\xi f \cdot \partial_x - \partial_x f \cdot \partial_\xi$, the Hamilton flow of a function $f \in C^\infty(T^*\mathbb{R}^n)$. For $\Phi_t = \text{Exp}(tH_p)$ taking the derivative with respect to the initial conditions gives a symplectomorphism $d_{x,\xi}\Phi_t(x, \xi) : T_{(x,\xi)}(T^*\mathbb{R}^n) \rightarrow T_{\Phi_t(x,\xi)}(T^*\mathbb{R}^n)$, and for $z_0 = (x_0, \xi_0)$ a critical point of p we have the fundamental automorphism

$$d_{x,\xi}\Phi_t(z_0) : T_{z_0}(T^*\mathbb{R}^n) \rightarrow T_{z_0}(T^*\mathbb{R}^n). \quad (6)$$

Near the critical point of p we can write

$$p(x, \xi) = E_c + \sum_{j=2}^N p_j(x, \xi) + \mathcal{O}(\|(x - x_0, \xi - \xi_0)\|^{N+1}),$$

where the functions p_j are homogeneous of degree j in $(x - x_0, \xi - \xi_0)$. In particular, p_2 is the Hessian in z_0 and can be interpreted as an invariantly defined quadratic form on $T_{z_0}(T^*\mathbb{R}^n)$.

Definition 1 For all $T \in \mathbb{R}$ let be $\mathfrak{F}_T = \text{Ker}(d_{x,\xi}\Phi_T(z_0) - \text{Id}) \subset T_{z_0}(T^*\mathbb{R}^n)$ and $\mathfrak{F}_T^\perp = T_{z_0}(T^*\mathbb{R}^n)/\mathfrak{F}_T$. To this linear subspace we associate its dimension

$l_T = 2d_T = \dim(\mathfrak{F}_T)$, and also the following three objects :

$$Q_T = p_2|_{\mathfrak{F}_T}, \quad (7)$$

$$\frac{1}{2}q_T = p_2|_{\mathfrak{F}_T^\perp}, \quad (8)$$

$$C_{Q_T} = \{(x, \xi) \in \mathfrak{F}_T / Q_T(x, \xi) = 0\}. \quad (9)$$

We say that $T \neq 0$ is a period of $d_{x,\xi}\Phi_t(z_0)$ if $\mathfrak{F}_T \neq \{0\}$ and that T is a **total period** of $d_{x,\xi}\Phi_t(z_0)$ when $\mathfrak{F}_T = T_{z_0}(T^*\mathbb{R}^n)$.

The next condition is inspired by proposition 2.1 of D. Khuat Duy [10] concerning Schrödinger operators and will be useful to separate the contributions of fixed points from those of the non-trivial periodic trajectories.

(H_5) For all period T of $d_{x,\xi}\Phi_t(z_0)$ there exists neighborhoods V_T of T and U_T of z_0 such that $\Phi_t(z) \neq z$ for all $T \in V_T$ and all $z \in U_T \setminus \{z_0\} \cap \Sigma_{E_c}$.

Note that (H_5) only concerns the dynamics on the energy surface Σ_{E_c} and Khuat Duy loc. cit. has shown that (H_5) is always satisfied by a Schrödinger-type Hamiltonian $\xi^2 + V(x)$. We give in the last section two examples of non-Schrödinger operators satisfying (H_5), to show that this class is non-empty. We note σ the usual symplectic form on $T_{z_0}(T^*\mathbb{R}^n)$ and by $d_z^l\Phi_t$ the derivative of order l , with respect to initial conditions, evaluated in (t, z) . If u is in a vector space V we use the notation $u^l = (u, \dots, u) \in V^l$.

Definition 2 If $k > 1$ is the first integer such that $d_{z_0}^k\Phi_T \neq 0$, we put

$$R_k(z) = \frac{1}{k!}\sigma(z, d_{z_0}^{k-1}\Phi_T(z^{k-1})), \quad z \in \mathfrak{F}_T, \quad (10)$$

and

$$\tilde{R}_k = R_k|_{C_{Q_T} \cap \mathbb{S}^{2d_T-1}}, \quad (11)$$

the restriction of R_k to the regular surface $C_{Q_T} \cap \mathbb{S}^{2d_T-1}$. Here \mathbb{S}^{2d_T-1} is the unit sphere, where we are working in local coordinates such that (5) holds. Finally, let dL_{Q_T} be the Liouville-measure on this surface, i.e. $dL_{Q_T} \wedge dQ_T = d\theta$ on $C_{Q_T} \cap \mathbb{S}^{2d_T-1}$, where $d\theta$ is the surface measure of \mathbb{S}^{2d_T-1} .

The derivatives $d_{z_0}^l\Phi_l$ are computed in section 4.4 and the relation $d_{z_0}^k\Phi_T \neq 0$ can be stated in terms of resonances, i.e. the arithmetical properties of the eigenvalues of p_2 or, more precisely, of Q_T .

If Θ is a cut-off function near the energy E_c we have

$$\gamma(E_c, h) = \text{Tr}(\varphi(\frac{P_h - E_c}{h})\Theta(P_h)) + \mathcal{O}(h^\infty), \quad (12)$$

as will show proposition 6 below. Hence, modulo $\mathcal{O}(h^\infty)$, we can write

$$\begin{aligned}\gamma(E_c, h) &= \text{Tr}(\psi_h^w \varphi(\frac{P_h - E_c}{h}) \Theta(P_h)) + \text{Tr}((1 - \psi_h^w) \varphi(\frac{P_h - E_c}{h}) \Theta(P_h)) \\ &= \gamma_1(z_0, E_c, h) + \gamma_2(z_0, E_c, h),\end{aligned}$$

where ψ is a function with compact support near z_0 such that (H_5) is valid on $\text{supp}(\psi)$ and the notation ψ_h^w stands for the classical Weyl h -quantization. Under the additional hypothesis of having a clean flow, the asymptotics of $\gamma_2(z_0, E_c, h)$ is given by the regular trace formula.

Without any loss of generality we can suppose that $\text{supp}(\hat{\varphi})$ is small enough near some non-zero period T of $d\Phi_t(z_0)$ such that T is the only period of $d\Phi_t(z_0)$ on $\text{supp}(\hat{\varphi})$, as follows by an easy partition of unity argument, the set of periods of the linearized flow being discrete. For the remaining contribution $\gamma_1(z_0, E_c, h)$ we then obtain

Theorem 3 *Under (H_1) to (H_5) and the assumption that Q_T is positive or negative definite, we have*

$$\gamma_1(z_0, E_c, h) = C(T) \Lambda_T(\varphi) + \mathcal{O}(h^{\frac{1}{2}}), \text{ as } h \rightarrow 0,$$

where

$$C(T) = -\frac{1}{2} \frac{\exp(i\frac{\pi}{4} \text{sgn}(q_T))}{|\det(q_T)|^{\frac{1}{2}}} \exp(i\pi \frac{d_T - 1}{2} \text{sign}(Q_T)) \Gamma(d_T),$$

and

$$\Lambda_T(\varphi) = \frac{1}{(2\pi)^{1+d_T}} \left\langle (t - T - i0)^{-d_T}, \frac{(t - T)^{d_T}}{|\det(\text{Id} - d\Phi_t(z_0))|^{\frac{1}{2}}} \hat{\varphi}(t) e^{itp^1(z_0)} \right\rangle.$$

In the case of a non definite Q_T we have :

Theorem 4 *If $n \geq 2$, under (H_1) to (H_5) and if $R_k(z) \neq 0$ for all $z \in C_{Q_T} \setminus \{0\}$, we have*

$$\gamma_1(z_0, E_c, h) = h^{\frac{2d_T+k-2}{k}-d_T} (K_T \hat{\varphi}(T) \exp(iTp^1(z_0)) + \mathcal{O}(h^{\frac{1}{k}})), \text{ for } h \rightarrow 0,$$

where

$$K_T = \mu_k(T) \exp(i\pi \frac{d_T - 1}{k} \text{sign}(\tilde{R}_k)) \int_{C_{Q_T} \cap \mathbb{S}^{2d_T-1}} |\tilde{R}_k(\theta)|^{-\frac{2d_T-2}{k}} dL_{Q_T}(\theta),$$

and

$$\mu_k(T) = -\frac{1}{k} \Gamma(\frac{2d_T - 2}{k}) \frac{\exp(i\frac{\pi}{4} \text{sgn}(q_T))}{|\det(q_T)|^{\frac{1}{2}} (2\pi)^{d_T+1}}.$$

More generally, if $\{z \in C_{Q_T} \setminus \{0\} : R_k(z) = 0\} \neq \emptyset$, but if $\nabla Q_T(z), \nabla R_k(z)$ are linearly independent on this set, then the same result holds with

$$K_T = \mu_k(T) \exp(i\pi \frac{d_T - 1}{k}) \int_{C_{Q_T} \cap \mathbb{S}^{2d_T-1}} (\tilde{R}_k(\theta) + i0)^{-\frac{2d_T-2}{k}} dL_{Q_T}(\theta).$$

Remark 5 Theorem 3 contains the class of Schrödinger operators $\xi^2 + V(x)$, with V non degenerate, since in this case $p_2|_{\text{Ker}(d\Phi_t(z_0) - \text{Id})}$ is positive definite. Moreover, the condition $n \geq 2$ in Theorem 4 is natural, since, under our assumptions, in dimension $n = 1$ we have $p_2(x_1, \xi_1) = \frac{1}{2}w_1(x_1^2 + \xi_1^2)$, or $\frac{1}{2}w_1(x_1^2 - \xi_1^2)$. In the first case Q_T is definite and in the second case 0 is the only period of the linearized flow.

3 Oscillatory representation of $\gamma(E_c, h)$

We introduce a cut-off function $\Theta \in C_0^\infty(]E_c - \varepsilon, E_c + \varepsilon[)$, such that $\Theta = 1$ near E_c and $0 \leq \Theta \leq 1$ on \mathbb{R} . We then have the decomposition

$$\begin{aligned} \gamma(E_c, h) &= \sum_{\lambda_j(h) \in I_\varepsilon} (1 - \Theta)(\lambda_j(h)) \varphi\left(\frac{\lambda_j(h) - E_c}{h}\right) + \sum_{\lambda_j(h) \in I_\varepsilon} \Theta(\lambda_j(h)) \varphi\left(\frac{\lambda_j(h) - E_c}{h}\right) \\ &= \gamma'(E_c, h) + \gamma''(E_c, h). \end{aligned} \tag{13}$$

Proposition 6 $\gamma'(E_c, h) = \mathcal{O}(h^\infty)$, when $h \rightarrow 0$.

Proof. With $\varphi \in \mathcal{S}(\mathbb{R})$ for all $k \in \mathbb{N}$ there exist C_k such that $|x^k \varphi(x)| \leq C_k$ on \mathbb{R} . If $N(h)$ is the number of eigenvalues inside $[E_c - \varepsilon, E_c + \varepsilon] \cap \text{supp}(1 - \Theta)$, then, by Theorem 3.13 of [15], we have the estimate

$$|\gamma'(E_c, h)| \leq N(h) C_k \left| \frac{\lambda_j(h) - E_c}{h} \right|^{-k}, \quad N(h) = \mathcal{O}(h^{-n}).$$

On the support of $(1 - \Theta)$ we have $|\lambda_j(h) - E_c| > \varepsilon_0 > 0$, this gives

$$|\gamma'(E_c, h)| \leq N(h) C_N \varepsilon_0^{-k} h^k \leq c_N h^{k-n}.$$

Since it is true for all $k \in \mathbb{N}$ the result follows. ■

As a consequence, the asymptotics of $\gamma(E_c, h)$, modulo $\mathcal{O}(h^\infty)$, is given by $\gamma''(E_c, h)$. Now, by Fourier transform, and inversion, we have that

$$\Theta(P_h)\varphi\left(\frac{P_h - E_c}{h}\right) = \frac{1}{2\pi} \int_{\mathbb{R}} \text{Exp}\left(\frac{it}{h}P_h\right)e^{-i\frac{tE_c}{h}}\hat{\varphi}(t)\Theta(P_h)dt.$$

The trace of the left-hand side is then exactly $\gamma''(E_c, h)$, and we can write

$$\gamma''(E_c, h) = \frac{1}{2\pi} \text{Tr} \int_{\mathbb{R}} \Theta(P_h) \text{Exp}\left(\frac{it}{h}P_h\right)e^{-i\frac{tE_c}{h}}\hat{\varphi}(t)dt. \quad (14)$$

The formula (14) uses the localized unitary group $U_{\Theta, h}(t) = \Theta(P_h)\text{Exp}\left(\frac{it}{h}P_h\right)$. A classical result, see e.g. [5], about this object is

Proposition 7 *Let Λ be the Lagrangian manifold associated to the flow of p*

$$\Lambda = \{(t, \tau, x, \xi, y, \eta) \in T^*\mathbb{R}^{2n+1} \mid \tau = -p(x, \xi), (x, \xi) = \Phi_t(y, \eta)\}, \quad (15)$$

then $U_{\Theta, h}(t)$ is a h -Fourier integral operator (or h -FIO) associated to Λ . More precisely, there exists for each N a FIO $U_{\Theta, h}^{(N)}(t)$ with integral kernel in the Hörmander class $I(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n, \Lambda)$ and operators $R_h^{(N)}(t)$ on $L^2(\mathbb{R}^n)$, with uniformly bounded norms for $0 < h \leq 1$ and t in a compact subset of \mathbb{R} , such that

$$U_{\Theta, h}(t) = U_{\Theta, h}^{(N)}(t) + h^N R_h^{(N)}(t). \quad (16)$$

We recall the compactly supported cut-off function $\psi = \psi(x, \xi)$, on whose support (H_5) holds. If ψ_1 is such that $\psi_1\psi = \psi$, with $\text{supp}(\psi_1)$ small enough, then by cyclicity of the trace,

$$\gamma_2(z_0, E_c, h) = \frac{1}{2\pi} \text{Tr} \int_{\mathbb{R}} \hat{\varphi}(t)\psi_h^w\Theta(P_h)\text{Exp}\left(\frac{it}{h}(P_h - E_c)\right)\psi_{1, h}^w dt + \mathcal{O}(h^\infty),$$

and $\psi_h^w\Theta(P_h)\text{Exp}\left(\frac{it}{h}(P_h - E_c)\right)\psi_{1, h}^w \in I(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n, \Lambda)$. After perhaps a local change of variable in y , the operator $\psi_h^w\Theta(P_h)\text{Exp}\left(\frac{it}{h}P_h\right)\psi_{1, h}^w$ can be approximated, modulo an error $\mathcal{O}(h^N)$, by an h -FIO with kernel

$$\frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(t, x, \xi) - \langle y, \xi \rangle)} b_N(t, x, y, \xi, h) d\xi,$$

see 4.1 below. Integrating this kernel on the diagonal gives, modulo terms of order $\mathcal{O}(h^N)$

$$\gamma_2(E_c, h) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}} \int_{T^*\mathbb{R}^n} e^{\frac{i}{h}(S(t, x, \xi) - \langle x, \xi \rangle + tE_c)} a_N(t, x, \xi, h) dt dx d\xi, \quad (17)$$

$$a_N(t, x, \xi, h) = \hat{\varphi}(t)b_N(t, x, x, \xi, h). \quad (18)$$

For a detailed construction we refer to [15] or [5].

Remark 8 Because of the presence of $\Theta(P_h)$ and $\hat{\varphi}$, the amplitudes a_N are of compact support. Since we are interested in the main contribution to the trace formula we note a the amplitude of (17), i.e. a depends on (h, N) .

4 Study of the phase function and of the classical dynamics

First, we study the nature of the critical points of (17). After we establish some results on the classical dynamics related to $p(x, \xi)$ and we compute the Taylor expansion of the phase function. Resonance-type conditions will naturally occur in the study of this question.

4.1 Singularity of the phase

By Theorem 5.3 of [9], we can, after perhaps a local change of variables in y , suppose that the flow Φ_t , near (x_0, ξ_0) and for $t \in \text{supp}(\hat{\varphi})$ sufficiently small, has a generating function $S(t, x, \eta)$, i.e.

$$(x, \xi) = \Phi_t(y, \eta) \Leftrightarrow \begin{cases} y = \partial_\eta S(t, x, \eta), \\ \xi = \partial_x S(t, x, \eta), \end{cases} \quad (19)$$

which therefore, by a classical result, satisfies the Hamilton-Jacobi equation $\partial_t S(t, x, \eta) + p(x, \partial_x S(t, x, \eta)) = 0$. Hence, near (T, x_0, ξ_0) , the Lagrangian manifold Λ of the flow is parameterized by the phase function $S(t, x, \eta) - \langle y, \eta \rangle$. This choice for the phase is only valid near (x_0, ξ_0) when $\xi_0 \neq 0$, but if $\xi_0 = 0$ we can change the operator P_h by $e^{\frac{i}{h}\langle x, \xi_1 \rangle} P_h e^{-\frac{i}{h}\langle x, \xi_1 \rangle}$ with $\xi_1 \neq 0$. This does not affect the spectrum and the new operator obtained has symbol $p(x, \xi - \xi_1)$ and critical point (x_0, ξ_1) . With (19) a critical point of (17) satisfies

$$\begin{cases} \partial_x S(t, x, \xi) = \xi, \\ \partial_\xi S(t, x, \xi) = x, \\ \partial_t S(t, x, \xi) = -E_c. \end{cases} \Leftrightarrow \begin{cases} \Phi_t(x, \xi) = (x, \xi), \\ p(x, \xi) = E_c. \end{cases}$$

The following lemma is classical and can for example be found in [11]. Recall that we will often denote points (x, ξ) of phase space by a single letter z .

Lemma 9 *Let us define $\Psi(t, x, \xi) = S(t, x, \xi) - \langle x, \xi \rangle + tE_c$, then if z_0 is critical point of Ψ , we have the equivalence $d_z^2 \Psi(z_0) \delta z = 0 \Leftrightarrow d_z \Phi_t(z_0) \delta z = \delta z$,*

$\forall \delta z \in T_{z_0}(T^*\mathbb{R}^n)$, i.e. the degenerate directions of the phase correspond to fixed points of the linearized flow at z_0 .

If we use Lemma 9, we obtain for our phase function

Corollary 10 *A critical point (T, x_0, ξ_0) of $\Psi(t, x, \xi)$ is degenerate with respect to (x, ξ) if and only if T is a period of the linearized flow $d_{x, \xi}\Phi_t(x_0, \xi_0)$.*

The next result is also well known from classical mechanics

Lemma 11 *If $\partial_x p(x_0, \xi_0) = \partial_\xi p(x_0, \xi_0) = 0$ then $d_{x, \xi}\Phi_t(x_0, \xi_0)$ is the Hamiltonian flow of the quadratic form $\frac{1}{2}d^2p(x_0, \xi_0)$ on $T_{x_0, \xi_0}(T^*\mathbb{R}^n)$.*

Let $T \neq 0$ be a period of $d\Phi_t(z_0)$. Corollary 10 shows that we must introduce

$$\Psi(t, x, \xi) = S(t, x, \xi) - \langle x, \xi \rangle = (t - T)g(t, x, \xi) + (S(T, x, \xi) - \langle x, \xi \rangle). \quad (20)$$

This function g is C^∞ and satisfies

$$g(T, x, \xi) = \frac{\partial S}{\partial t}(T, x, \xi) = -p(x, \partial_x S(T, x, \xi)).$$

To simplify the notations we write $g(t, z) = g(t, x, \xi)$ and

$$R(z) = R(x, \xi) = S(T, x, \xi) - \langle x, \xi \rangle. \quad (21)$$

Lemma 12 *In a neighborhood of z_0 , and near T , the only critical point, on the energy surface Σ_{E_c} , of the functions $S(T, x, \xi) - \langle x, \xi \rangle$ and $g(t, x, \xi)$ is z_0 .*

Proof. First, $d_{x, \xi}(S(T, x, \xi) - \langle x, \xi \rangle) = 0$ is equivalent to $\Phi_T(x, \xi) = (x, \xi)$. But near z_0 and with (H_5) this can only be satisfied for $(x, \xi) = z_0$.

Next, we consider $g(T, x, \xi) = -p(x, \partial_x S(T, x, \xi))$. here $d_{x, \xi}g(T, x, \xi) = 0$ is equivalent to

$$\begin{cases} \partial_x p(x, \partial_x S(T, x, \xi)) + \partial_\xi p(x, \partial_x S(T, x, \xi))(\partial_{x, x}^2 S(T, x, \xi)) = 0 \\ \partial_\xi p(x, \partial_x S(T, x, \xi))(\partial_{x, \xi}^2 S(T, x, \xi)) = 0. \end{cases}$$

Clearly z_0 is critical since $\partial_x S(T, x_0, \xi_0) = \xi_0$. If z is a critical point of $g(T, x, \xi)$ we have

$$\begin{pmatrix} \text{Id}_n & \partial_{x, x}^2 S(T, x, \xi) \\ 0 & \partial_{x, \xi}^2 S(T, x, \xi) \end{pmatrix} \begin{pmatrix} \partial_x p(x, \partial_x S(T, x, \xi)) \\ \partial_\xi p(x, \partial_x S(T, x, \xi)) \end{pmatrix} = 0. \quad (22)$$

Where I_n is the identity matrix of order n . But $\Phi_t(\partial_\xi S(t, x, \xi), \xi) = (x, \partial_x S(t, x, \xi))$ and this leads to

$$d_{x,\xi}\Phi_t(\partial_\xi S(t, x, \xi), \xi) \begin{pmatrix} \partial_{\xi,x}^2 S(t, x, \xi) & \partial_{\xi,\xi}^2 S(t, x, \xi) \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ \partial_{x,x}^2 S(t, x, \xi) & \partial_{x,\xi}^2 S(t, x, \xi) \end{pmatrix}.$$

Since $d\Phi_t$ is an isomorphism Eq.(22) imposes

$$\partial_x p(x, \partial_x S(T, x, \xi)) = \partial_\xi p(x, \partial_x S(T, x, \xi)) = 0.$$

In a suitable neighborhood of z_0 this implies that $(x, \xi) = z_0$, since z_0 is non-degenerate. With $(t - T)g(t, z) = S(t, z) - S(T, z)$, for $t \neq T$, the critical points of $g(t, z)$ are those of $S(t, z) - S(T, z)$, and equation (19) shows that

$$\begin{aligned} \Phi_t(\partial_\xi S(t, x, \xi), \xi) &= (x, \partial_x S(t, x, \xi)) = (x, \partial_x S(T, x, \xi)) \\ &= \Phi_T(\partial_\xi S(T, x, \xi), \xi) = \Phi_t(\partial_\xi S(T, x, \xi), \xi). \end{aligned}$$

We then have by the group law

$$\Phi_{-t}(\Phi_T(\partial_\xi S(T, x, \xi), \xi)) = \Phi_{T-t}(\partial_\xi S(t, x, \xi), \xi) = (\partial_\xi S(t, x, \xi), \xi).$$

For $t \neq T$ and $|t - T|$ small the point $(\partial_\xi S(t, x, \xi), \xi)$ would be periodic, with period $(t - T)$. For $(x, \xi) \in \Sigma_{E_c}$, and near z_0 , (H_5) implies that $(x, \xi) = z_0$. ■

The Hessian matrix with respect to $z = (x, \xi)$ of g in (T, z_0) satisfies

$$\left\{ \begin{aligned} \langle \text{Hess}_z(g)(T, z_0)(\delta x, \delta \xi), (\delta x, \delta \xi) \rangle &= \langle \text{Hess}(p)(z_0)B(\delta x, \delta \xi), B(\delta x, \delta \xi) \rangle, \\ B &= \begin{pmatrix} I_n & 0 \\ \partial_{x,x}^2 S(T, z_0) & \partial_{x,\xi}^2 S(T, z_0) \end{pmatrix}, \end{aligned} \right. \quad (23)$$

with B non singular, as seen before. This proves the relation

$$\text{Hess}_z(g)(T, z_0) = {}^t B \text{Hess}(p)(z_0) B. \quad (24)$$

In the case of a total period T of $d\Phi_t(z_0)$ we obtain

Proposition 13 *If $d\Phi_t(z_0)$ is totally periodic, with period T , then the function $g(t, x, \xi)$ satisfies $\text{Hess}_z(g)(T, z_0) = \text{Hess}(p)(z_0)$.*

4.2 The linearized flow in z_0

Up to a permutation of coordinates we can assume that

$$p_2(x, \xi) = \frac{1}{2} \left(\sum_{j=1}^k w_j (x_j^2 + \xi_j^2) + \sum_{j=k+1}^n w_j (x_j^2 - \xi_j^2) \right). \quad (25)$$

The flow of p_2 , viewed as an element of $\text{End}(T_0(T^*\mathbb{R}^n)) \simeq \text{End}(\mathbb{R}^{2n})$, is

$$\text{Exp}(tH_{p_2})(x, \xi) = A(t) \begin{pmatrix} x \\ \xi \end{pmatrix}, \quad A(t) = \begin{pmatrix} a(t) & 0 & c(t) & 0 \\ 0 & b(t) & 0 & d(t) \\ -c(t) & 0 & a(t) & 0 \\ 0 & d(t) & 0 & b(t) \end{pmatrix},$$

where $(x, \xi) = (x', x'', \xi', \xi'')$, $x', \xi' \in \mathbb{R}^k$, $x'', \xi'' \in \mathbb{R}^{n-k}$, and

$$\begin{cases} a(t) = \text{diag}(\cos(w_i t)), & b(t) = \text{diag}(\text{ch}(w_i t)), \\ c(t) = \text{diag}(\sin(w_i t)), & d(t) = \text{diag}(\text{sh}(w_i t)), \end{cases}$$

and "diag" means diagonal matrix. In the following we work on the subspace $\{x'' = \xi'' = 0\}$ obtained by projection on the periodic variables. Let be I a subset of $\{1, \dots, n\}$ with l elements, $l > 1$. The existence of a non trivial closed trajectory of dimension l imposes that there exists a $c \in \mathbb{R}^*$ such that

$$\forall i \in I, \exists n_i \in (\mathbb{Z}^*)^l : w_i = c n_i. \quad (26)$$

Remark 14 Let $M(w) = \{k \in \mathbb{Z}^n / \langle k, w \rangle = 0\}$ be the \mathbb{Z} -module of resonances of the vector $w = (w_1, \dots, w_n)$. The relations (26), for $l > 1$ lead to resonances, since $n_i w_i - n_j w_j = 0$, but a resonant system can have no periodic trajectories of dimension greater than 1, as is shown by $(\sqrt{2} + \sqrt{3}, \sqrt{2} - \sqrt{3}, \sqrt{2})$.

Since we are interested by the periods of $d\Phi_t(z_0)$ we define (assuming (25))

$$Q_{\text{per}}(x, \xi) = \frac{1}{2} \sum_{j=1}^k w_j (x_j^2 + \xi_j^2),$$

and also

$$\begin{aligned} Q^+(x, \xi) &= \sum_{j=1}^{k_1} \frac{w_j}{2} (x_j^2 + \xi_j^2), \quad w_i > 0, \\ Q^-(x, \xi) &= \sum_{j=k_1+1}^k \frac{|w_j|}{2} (x_j^2 + \xi_j^2), \quad w_i < 0, \end{aligned}$$

so that $Q_{per}(x, \xi) = Q^+(x, \xi) - Q^-(x, \xi)$. Let P_1 and P_2 be the linear subspaces obtained by projecting orthogonally on the effective variables of Q^+ , Q^- .

Proposition 15 *If we have $\mathfrak{F}_T \subset P_1$ or $\mathfrak{F}_T \subset P_2$, then $\text{Hess}(g)(T, z_0)|_{\mathfrak{F}_T}$ is respectively positive or negative definite.*

Proof. Choosing coordinates as in (25), we see that for $v \in \mathfrak{F}_T$

$$A(T) \begin{pmatrix} \partial_{x,\xi}^2 S(T, z_0) & \partial_{\xi,\xi}^2 S(T, z_0) \\ 0 & I_n \end{pmatrix} v = \begin{pmatrix} I_n & 0 \\ \partial_{x,x}^2 S(T, z_0) & \partial_{x,\xi}^2 S(T, z_0) \end{pmatrix} v.$$

With $\dim(\mathfrak{F}_T) = 2d_T$ we can choose our coordinates such that

$$\begin{pmatrix} a(T) & 0 \\ 0 & b(T) \end{pmatrix} \partial_{x,\xi}^2 S(T, z_0) = \begin{pmatrix} \begin{pmatrix} I_{d_T} & 0 \\ 0 & * \end{pmatrix} & 0 \\ 0 & b(T) \end{pmatrix} \partial_{x,\xi}^2 S(T, z_0) = I_n.$$

Elementary considerations show that

$$\partial_{x,\xi}^2 S(T, z_0) = \begin{pmatrix} I_{d_T} & 0 \\ 0 & * \end{pmatrix}, \quad \partial_{x,x}^2 S(T, z_0) = \begin{pmatrix} 0_{d_T} & 0 \\ 0 & * \end{pmatrix}, \quad (27)$$

where $*$ designs matrix blocs which are irrelevant for the present discussion. Hence, by restriction to $\mathfrak{F}_T : (\text{Hess}(g)(T, z_0))|_{\mathfrak{F}_T} = (\text{Hess}(p)(z_0))|_{\mathfrak{F}_T}$. \blacksquare

4.3 Taylor series of the flow near z_0

We start by the general case of an autonomous system near an equilibrium. Let be Φ_t the flow of a C^∞ vector field X on \mathbb{R}^n with coordinates $z = (z_1, \dots, z_n)$ and let z_0 a fixed point of Φ_t . We denote by $A(z_0)$ the matrix of the linearization of X in z_0

$$A(z_0) = \left(\frac{\partial X^i}{\partial z_k} \right)_{(i,k)}(z_0), \quad (28)$$

and we recall that $d\Phi_t(z_0) = \text{Exp}(tA(z_0))$. Here, and in the following, the derivatives d will be taken with respect to z , we denote by $d^k f$ the k -th derivative of f regarded as a multi-linear form on the k -fold product $\mathbb{R}^n \times \dots \times \mathbb{R}^n$, and $d^k f(z_0)$ or $d_{z_0}^k f$ this derivative evaluated in z_0 . As an example,

we compute the second derivative of the flow in z_0 :

$$\begin{aligned} \partial_{z_i, z_j}^2 \left(\frac{d}{dt} \Phi_t(z) \right) &= \sum_{k, l=1}^n \frac{\partial^2 X}{\partial z_k \partial z_l}(\Phi_t(z)) \frac{\partial \Phi_t^l(z)}{\partial z_j} \frac{\partial \Phi_t^k(z)}{\partial z_i} \\ &\quad + \sum_{k=1}^n \frac{\partial X}{\partial z_k}(\Phi_t(z)) \frac{\partial^2 \Phi_t^k(z)}{\partial z_i \partial z_j}. \end{aligned} \quad (29)$$

Hence at the point z_0 we obtain

$$\frac{d}{dt}(\partial_{z_i, z_j}^2 \Phi_t(z_0)) = \sum_{k, l=1}^n \frac{\partial^2 X}{\partial z_k \partial z_l}(z_0) \frac{\partial \Phi_t^l(z_0)}{\partial z_j} \frac{\partial \Phi_t^k(z_0)}{\partial z_i} + A(z_0) \partial_{z_i, z_j}^2 \Phi_t(z_0).$$

Let us write $\text{Hess}(X)(z_0)$ for the vector valued Hessian of X evaluated in z_0 . Interpreting (29) as an inhomogeneous system of equations for $\partial_{z_i, z_j}^2 \Phi_t(z_0)$ we obtain, since $\Phi_0 = \text{Id}$ and therefore $d_{z, z}^2 \Phi_0(z_0) \equiv 0$, that

$$d^2 \Phi_t(z_0)(z, z) = d\Phi_t(z_0) \int_0^t d\Phi_{-s}(z_0) \text{Hess}(X)(z_0)(d\Phi_s(z_0)(z), d\Phi_s(z_0)(z)) ds. \quad (30)$$

Now, let us assume that z_0 is the origin, we generalize as follows :

Proposition 16 *If $X(z) = Az + X_k(z) + \mathcal{O}(\|z\|^{k+1})$, where $X_k(z)$ is homogeneous of degree $k > 2$ in z , then the flow Φ_t of X satisfies*

$$\begin{aligned} d^j \Phi_t(0) &= 0, \quad \forall j \in \{2, \dots, k-1\}, \\ d^k \Phi_t(0)(z, \dots, z) &= d\Phi_t(0) \int_0^t d\Phi_{-s}(0) d^k X(0)(d\Phi_s(0)(z), \dots, d\Phi_s(0)(z)) ds. \end{aligned}$$

Proof. The first result is trivial. At the order k we have for $|\alpha| = k$

$$\frac{d}{dt} \left(\left(\frac{\partial}{\partial z} \right)^\alpha \Phi_t(z) \right) = \left(\frac{\partial}{\partial z} \right)^\alpha (X(\Phi_t(z))).$$

Hence, for $z = 0$ we simply have

$$\frac{d}{dt} \left(\left(\frac{\partial}{\partial z} \right)^\alpha \Phi_t(z) \right)_{|z=0} = d^\alpha X(0)(d\Phi_t(z_0), \dots, d\Phi_t(z_0)) + A \left(\left(\frac{\partial}{\partial z} \right)^\alpha \Phi_t(z) \right)_{|z=0}. \quad (31)$$

Eq. (31) is linear of the form $d_t u_\alpha(t) = f_\alpha(t) + A u_\alpha(t)$, and by integration, with the initial condition $d^k \Phi_0(0) = 0$, we obtain

$$\left(\left(\frac{\partial}{\partial z} \right)^\alpha \Phi_t \right)(0) = d\Phi_t(0) \int_0^t d\Phi_{-s}(0) d^\alpha X(0)(d\Phi_s(0), \dots, d\Phi_s(0)) ds,$$

and the result holds by linearity. ■

A more general result is

Theorem 17 *Let be z_0 an equilibrium point of X and Φ_t the flow of X . For all $m \in \mathbb{N}^*$, there exists a polynomial map P_m , of degree at most m , such that*

$$d^m \Phi_t(z_0)(z^m) = d\Phi_t(z_0) \int_0^t d\Phi_{-s}(z_0) P_m(d\Phi_s(z_0)(z), \dots, d^{m-1} \Phi_s(z_0)(z^{m-1})) ds. \quad (32)$$

In addition P_m is uniquely determined by the m -jet of X in z_0 .

Proof. For $m = 1$, $d\Phi_t(z_0)$ is determined by the operator $A(z_0)$, i.e. by the 1-jet of X . We note $x^l \in (\mathbb{R}^n)^l$ the image of x under the diagonal mapping, with the same convention for any vector. If f and g are smooth we obtain

$$d^m(fog)(x_0)(x^m) = \sum_{j=1}^m \sum_{\alpha \in I_j} c_\alpha d^j f(g(x_0))((dg(x_0)(x))^{\alpha_{j1}}, \dots, (d^m g(x_0)(x^m))^{\alpha_{jm}}),$$

with $c_\alpha \in \mathbb{N}$, $I_j = \{\alpha \in \mathbb{N}^m / \sum_{k=1}^m k\alpha_{j_k} = j\}$. Since z_0 is a fixed point, we have

$$d^m(X \circ \Phi_t)(z_0)(z^m) = \sum_{j=1}^m \sum_{\alpha \in I_j} c_\alpha d^j X(z_0)((d\Phi_s(z_0)(z))^{\alpha_{j1}}, \dots, (d^m \Phi_s(z_0)(z^m))^{\alpha_{jm}}).$$

For $Y = (Y_1, \dots, Y_m)$, we can define

$$P_m(Y) = \sum_{j=1}^m \sum_{\alpha \in I_j} c_\alpha d^j X(z_0)(Y_1^{\alpha_{j1}}, \dots, Y_m^{\alpha_{jm}}) - dX(z_0)(Y_m), \quad (33)$$

this leads to the differential equation, operator valued

$$\frac{d}{dt}(d^m \Phi_t(z_0))(z^m) = A(z_0) d^m \Phi_t(z_0)(z^m) + P_m(d\Phi_s(z_0)(z), \dots, d^{m-1} \Phi_s(z_0)(z^{m-1})).$$

With the initial condition $d^m \Phi_0(z_0) = 0$, we obtain that the solution is given by (32). Moreover, Eq.(33) shows that P_m is completely determined by the derivatives of order less or equal than m of X . ■

4.4 Application to Hamiltonian systems

Proposition 16 applied to the flow of H_p shows that

$$d^2 \Phi_t(z_0)(z, z) = d\Phi_t(z_0) \int_0^t d\Phi_{-s}(z_0) \text{Hess}(H_p)(z_0)(d\Phi_s(z_0)(z), d\Phi_s(z_0)(z)) ds.$$

We now consider more closely $d^2\Phi_t(z_0)$ for $t = T$, a period of $d\Phi_t(z_0)$. We introduce the following terminology

Definition 18 $w \in (\mathbb{R}^*)^n$ is pseudo-resonant to the order $l \in \mathbb{N}$ if

$$\exists (i_1, \dots, i_l) \in \{1, \dots, n\}, (\varepsilon_{i_1}, \dots, \varepsilon_{i_l}) \in \{-1, 1\} \text{ such that } \sum_{j=1}^l \varepsilon_{i_j} w_{i_j} = 0. \quad (34)$$

Remark 19 This notion is weaker than the usual resonance condition since for l even there always exists a pseudo-resonance. For example, for $l = 4$ we have $(w_i - w_i) \pm (w_j - w_j) = 0$, although w can be non-resonant at the order 4. We also observe that all resonances of order 3 are pseudo-resonances.

In term of the frequencies w_i , we then have

Theorem 20 *If the frequencies w satisfy no pseudo-resonance relation of order 3 and if T is a total period of $d\Phi_t(z_0)$, we have $d^2\Phi_T(z_0) = 0$.*

Proof. Under the condition (H_3) , $d^2\Phi_t(z_0)$ can be expressed as a linear combination of integrals of the elementary functions $s \mapsto \exp(\pm is(w_i + \varepsilon_1 w_j + \varepsilon_2 w_k))$ with $\varepsilon_j = \pm 1$. Hence, to determine $d^2\Phi_T(z_0)$ we must compute

$$\int_0^T \exp(\pm is(w_i + \varepsilon_1 w_j + \varepsilon_2 w_k)) ds,$$

but under the assumptions of the theorem all these integrals are 0. ■

For all $l \in \mathbb{N}^*$ let be $M_l(W) = \{k \in \mathbb{Z}^{2n} / \langle k, (w, w) \rangle = 0, |k| = l\}$ the \mathbb{Z} -module of resonances of order l . In the presence of resonances we can say the following :

Proposition 21 *For $k \in M_3(W)$ let be $\tilde{k} = (|k_1|, \dots, |k_{2n}|)$. If $\forall k \in M_3(W)$ we have $\frac{\partial^3 p}{\partial z^{\tilde{k}}}(z_0) = 0$ and if T is a total period of $d_{z_0}\Phi_t$ then $d^2\Phi_T(z_0) = 0$.*

The proof is trivial when going back to the proof of Theorem 20. ■

Let f^* be the pullback by a map f . Then by proposition 16 we have :

Corollary 22 *For a Hamiltonian system with the equilibrium point z_0 and such that $d_{z_0}^j p = 0, \forall j \in \{3, \dots, k-1\}$, we obtain*

$$d^j\Phi_t(z_0) = 0, \forall t, \forall j \in \{2, \dots, k-2\},$$

$$d_{z_0}^{k-1}\Phi_t(z^{k-1}) = d\Phi_t(z_0) \int_0^t d\Phi_{-s}(z_0) \left(d\Phi_s(z_0)^* (d_{z_0}^{k-1} H_p) \right) (z^{k-1}) ds.$$

And Theorem 20 generalizes trivially to the order k under the conditions of Corollary 22. More precisely, if k is odd and if there is no pseudo-resonance of order k then, under the assumptions of Corollary 22, we have $d_{z_0}^{k-1}\Phi_t = 0$.

4.5 Relation between the phase and the flow

Like in the preceding section we consider a Hamiltonian function p with total period T for the linearized flow, satisfying, near 0

$$p(z) = E_c + p_2(z) + \mathcal{O}(\|z\|^k). \quad (35)$$

We recall that $z = (x, \xi)$ and $z^k = (z, \dots, z) \in \mathbb{R}^{2nk}$. By Taylor, we have

$$\Phi_T(z) = z + \frac{1}{(k-1)!} d^{k-1}\Phi_T(0)(z^{k-1}) + \mathcal{O}(\|z\|^k).$$

Under these conditions we can write the generating function at time T as

$$S(T, x, \xi) = \langle x, \xi \rangle + R_k(x, \xi) + R_{k+1}(x, \xi), \quad (36)$$

where R_k is homogeneous of degree k and R_{k+1} is the remainder of the Taylor expansion. Let J be the matrix of the standard symplectic form on $T^*\mathbb{R}^n$. The relation between the phase function and the flow of p is given by

Proposition 23 *Under conditions (35) the $(k-1)$ -st derivative of the flow Φ_t , at time $t = T$, equals :*

$$d^{k-1}\Phi_T(0)((x, \xi)^{k-1}) = -(k-1)! J \nabla R_k(x, \xi) \quad (37)$$

In addition we have $R_j(x, \xi) = 0$, $3 \leq j < k$, and

$$R_k(x, \xi) = \frac{1}{k!} \int_0^T \sigma((x, \xi), d\Phi_{-s}(0) \left(d\Phi_s(0)^* (d_0^{k-1} H_p) \right) (x, \xi)^{k-1}) ds. \quad (38)$$

Proof. With equation (19) we obtain

$$\Phi_T(x + \partial_\xi R_k + \partial_\xi R_{k+1}, \xi) = (x + \partial_\xi R_k, \xi) + \frac{1}{(k-1)!} d^{k-1}\Phi_T(0)((x, \xi)^{k-1}) + \mathcal{O}(\|(x, \xi)\|^k),$$

by identification of homogeneous terms we have successively

$$\begin{aligned} (-\partial_\xi R_k, \partial_x R_k)(x, \xi) &= -J \nabla R_k(x, \xi) = \frac{1}{(k-1)!} d^{k-1}\Phi_T(0)((x, \xi)^{k-1}), \\ R_k(x, \xi) &= \frac{1}{k!} \langle (x, \xi), -J \nabla R_k(x, \xi) \rangle = \frac{1}{k!} \sigma((x, \xi), d^{k-1}\Phi_T(0)((x, \xi)^{k-1})), \end{aligned}$$

where the last result holds by homogeneity. Corollary 22 then implies (38). ■

5 Normal forms of the phase function

In this section we derive suitable normal forms for $\Psi(t, x, \xi) = S(t, x, \xi) - \langle x, \xi \rangle$ in our oscillatory integral representation of $\gamma_2(E_c, h)$. We recall the decomposition

$$\Psi(t, x, \xi) = (t - T)g(t, x, \xi) + R(x, \xi),$$

cf. formulas (20) and (21). In the micro-local neighborhood of $z_0 = (x_0, \xi_0)$ we are interested in, the only critical point of R and of $g(t, \cdot)$ is z_0 for t close to T and, moreover, z_0 is non-degenerate for the latter.

A further very important simplifying assumption we will make for the moment is that, until further notice, T is a total period of $d\Phi_t(z_0)$. We will show in section 6.3 below how to relax this assumption. If T is such a total period, then clearly $R(z) = \mathcal{O}(\|z\|^3)$. This can be made more precise

Lemma 24 *If near z_0 the function p satisfies (H_3) and condition (35) then for t near T there exist a non degenerate quadratic form $Q_t(x, \xi)$ such that $Q_T(x, \xi) = p_2(x, \xi)$ and*

$$S(t, x, \xi) - \langle x, \xi \rangle = (t - T) (Q_t(x, \xi) + h(t, x, \xi)) + R(x, \xi),$$

with $R(x, \xi) = \mathcal{O}(\|(x, \xi)\|^k)$ and $h(t, x, \xi) = \mathcal{O}(\|(x, \xi)\|^k)$ uniformly in t .

Proof. On replacing $t - T$ by t we can write $\Psi = R(z) + tG(t, z)$ with $G(t, z) = g(t + T, z)$. By a second order Taylor expansion around z_0 and proposition 13 we have that $G(t, z) = Q_t(z) + h(t, z)$, with $Q_0(z) = p_2(z)$ and $h(t, z) = \mathcal{O}(\|z\|^3)$. Now by proposition 23, $R(z) = \mathcal{O}(\|z\|^k)$, given that p satisfies (35), and since $\Phi_T(z) = z + \mathcal{O}(\|z\|^{k-1})$, we have that $S(T, x, \xi) = \langle x, \xi \rangle + \mathcal{O}(\|(x, \xi)\|^k)$. Therefore $h(t, z) = \mathcal{O}(\|z\|^k)$ uniformly in t and the lemma follows. ■

Reduction of the phase with respect to C_{Q_T} .

Let us suppose $S(T, x, \xi)$ contains effectively some terms of order k . We write, as before, $R(z) = R_k(z) + R_{k+1}(z)$, where R_k is the homogeneous component of degree k of R and R_{k+1} is the remainder of the Taylor series. The following lemma is useful for any perturbation by a function of odd degree.

Lemma 25 *Let be Q a non degenerate quadratic form on \mathbb{R}^n , $n > 3$, with inertia indices greater than 2. For all odd continuous function R , Q and R have a common zero on \mathbb{S}^{n-1} .*

Proof. Up to a linear change of coordinates we can assume that $Q(x) = \|x_1\|^2 - \|x_2\|^2$, with $x_1 \in \mathbb{R}^p$, $x_2 \in \mathbb{R}^q$, $p, q \geq 2$, $p + q = n$. Now the cone of the zeros of Q is invariant under isometries of the subspaces $(x_1, 0)$ and $(0, x_2)$. By rotating around the origins there exist a continuous curve γ_1 inside the

cone of Q mapping (x_1, x_2) to $(-x_1, x_2)$ and a curve γ_2 mapping $(-x_1, x_2)$ to $(-x_1, -x_2)$, inside the cone. If $\gamma = \gamma_1 \cdot \gamma_2$ is the union of the two previous curves, the function $R(\gamma)$ gives the result by continuity since R is odd. ■

Remark 26 A consequence of Lemma 25 is that the set $C_{Q_T} \cap C_{R_k} \cap \mathbb{S}^{n-1}$ is not empty when the function R_k is non-zero and odd and Q_T is non-definite.

We choose polar coordinates $z = (x, \xi) = r\theta$, $\theta \in \mathbb{S}^{2n-1}(\mathbb{R})$. These coordinates will perform a "blow-up" of $\mathbb{R} \times (T^*\mathbb{R}^n \setminus \{0\})$. In general one uses the projective space $\mathbb{P}_{2n-1}(\mathbb{R})$, but here, since the singularities are carried by the conic set of the zeros of Q_T , it is convenient to use the sphere \mathbb{S}^{2n-1} . For any function f , positively homogeneous on \mathbb{R}^n , we note $C_f = \{x \in \mathbb{R}^n / f(x) = 0\}$, the conic set of the zeros of f . Finally, $g \simeq h$ means that applications g and h are conjugated by a local diffeomorphism.

Lemma 27 *If $\theta_0 \in \mathbb{S}^{2n-1}$ with $\theta_0 \notin C_{Q_0}$, there exists a system of local coordinates χ , near $(t, r, \theta) = (0, 0, \theta_0)$, such that $\Psi \simeq \chi_0 \chi_1^2$ in a neighborhood of $(\chi_0, \chi_1) = (0, 0)$.*

Proof. Using the notations of Lemma 24 we have, in polar coordinates,

$$\Psi(t, z) \simeq r^2(tQ_t(\theta) + tr^{k-2}h_1(t, r, \theta) + r^{k-2}R_k(\theta) + r^{k-1}\tilde{R}(r, \theta)),$$

with $\tilde{R} \in C^\infty(\mathbb{R}_+ \times \mathbb{S}^{2n-1})$ and $h_1 \in C^\infty(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}^{2n-1})$. We choose new coordinates

$$\begin{aligned} \chi_0(t, r, \theta) &= tQ_t(\theta) + r^{k-2}(th_1(t, r, \theta) + R_k(\theta) + r\tilde{R}(r, \theta)), \\ (\chi_1, \chi_2, \dots, \chi_{2n})(t, r, \theta) &= (r, \theta). \end{aligned}$$

Hence $|\frac{D\chi}{D(t, r, \theta)}|(0, 0, \theta_0) = |Q_0(\theta_0)| \neq 0$ and in the new system of coordinates we have $\Psi(t, z) = (\chi_0 \chi_1^2)(t, r, \theta)$ as required. ■

Lemma 28 *If $\theta_0 \in C_{Q_0}$ with $\theta_0 \notin C_{R_k}$ there exists a system of local coordinates χ , near $(t, r, \theta) = (0, 0, \theta_0)$, such that $\Psi \simeq \chi_0 \chi_2 \chi_1^2 \pm \chi_1^k$ in a neighborhood of $(\chi_0, \chi_1, \chi_2) = (0, 0, 0)$.*

Proof. As in Lemma 27 we write

$$\Psi(t, z) = tr^2(Q_t(\theta) + r^{k-2}h_1(t, r, \theta)) + r^k(R_k(\theta) + r\tilde{R}(r, \theta)).$$

Since $R_k(\theta_0) \neq 0$ in a suitable neighborhood of $(\theta_0, 0)$ we have $R_k(\theta) + r\tilde{R}(r, \theta) \neq 0$. Up to a permutation of the variables θ_i , we can suppose that

$\frac{\partial Q_0}{\partial \theta_1}(\theta_0) \neq 0$. In a neighborhood of $(t, r, \theta) = (0, 0, \theta_0)$, we choose coordinates

$$\begin{aligned}\chi_0(t, r, \theta) &= t|R_k(\theta) + r\tilde{R}(r, \theta)|^{-\frac{2}{k}}, \\ \chi_1(t, r, \theta) &= r|R_k(\theta) + r\tilde{R}(r, \theta)|^{\frac{1}{k}}, \\ \chi_2(t, r, \theta) &= Q_t(\theta) + r^{k-2}h_1(t, r, \theta), \\ (\chi_3, \dots, \chi_{2n})(t, r, \theta) &= (\theta_2, \dots, \theta_{2n-1}).\end{aligned}$$

The corresponding Jacobian, at the point $(0, 0, \theta_0)$, is

$$|\frac{D\chi}{D(t, r, \theta)}(0, 0, \theta_0)| = |R_k(\theta_0)|^{-\frac{1}{k}}|\frac{\partial Q_0}{\partial \theta_1}(\theta_0)| \neq 0. \quad (39)$$

In these new coordinates the phase is $\Psi(t, r, \theta) = (\chi_0\chi_2\chi_1^2 + \chi_1^k)(t, r, \theta)$. \blacksquare

Lemma 29 *If $\theta_0 \in C_{Q_0} \cap C_{R_k}$ and if $\nabla Q_0(\theta_0), \nabla R_k(\theta_0)$ are linearly independent there exists a system of local coordinates χ near $(t, r, \theta) = (0, 0, \theta_0)$ such that $\Psi \simeq \chi_0\chi_1^2\chi_2 \pm \chi_1^k\chi_3$ in a neighborhood of $(\chi_0, \chi_1, \chi_2, \chi_3) = (0, 0, 0, 0)$.*

Proof. Near θ_0 we can complete $(Q_0(\theta), R_k(\theta))$ to a system of coordinates on the sphere. Up to a permutation of the θ_i , we can choose

$$\begin{aligned}(\chi_0, \chi_1)(t, r, \theta) &= (t, r), \\ \chi_2(t, r, \theta) &= Q_t(\theta) + rh_1(t, r, \theta), \\ \chi_3(t, r, \theta) &= R_k(\theta) + r\tilde{R}(r, \theta), \\ (\chi_4, \dots, \chi_{2n})(t, r, \theta) &= (\theta_3, \dots, \theta_{2n-1}).\end{aligned}$$

Then the corresponding Jacobian is

$$|\frac{D\chi}{D(t, r, \theta)}|(0, 0, \theta_0) = \left| \begin{pmatrix} \frac{\partial \chi_2}{\partial \theta_1} & \frac{\partial \chi_2}{\partial \theta_2} \\ \frac{\partial \chi_3}{\partial \theta_1} & \frac{\partial \chi_3}{\partial \theta_2} \end{pmatrix} \right| (0, 0, \theta_0) = \left| \begin{pmatrix} \frac{\partial Q_0}{\partial \theta_1} & \frac{\partial Q_0}{\partial \theta_2} \\ \frac{\partial R_k}{\partial \theta_1} & \frac{\partial R_k}{\partial \theta_2} \end{pmatrix} \right| (\theta_0) \neq 0.$$

In these new coordinates we have $\Psi(t, z) = (\chi_0\chi_1^2\chi_2 + \chi_1^k\chi_3)(t, r, \theta)$. \blacksquare

Remark 30 Coordinates χ form a system of admissible charts near (T, z_0) and these are singular in $z = z_0$ as coordinates on $T^*\mathbb{R}^n$. In the three systems of coordinates the measures are $r^{2n-1}|\frac{D\chi}{D(t, r, \theta)}(t, r, \theta)|dtdrdd\theta$, this term r^{2n-1} plays a major role since the critical sets of our normal forms are $\{r = 0\}$.

Combining Lemma 27, 28 and 29 gives

Theorem 31 *If (H_3) and conditions of Lemma 29 are satisfied and if T is a total period of $d\Phi_t(z_0)$, the phase function $S(t, x, \xi) - \langle x, \xi \rangle + tE_c$ has one of*

the following normal forms on the blow-up of (T, x_0, ξ_0) :

$$\text{first normal forms} : (\pm \chi_0 \chi_1^2) \text{ near } (U \setminus C_{Q_0}), \quad (40)$$

$$\text{second normal forms} : (\chi_0 \chi_1^2 \chi_2 \pm \chi_1^k) \text{ near } (C_{Q_0} \setminus C_{R_k}), \quad (41)$$

$$\text{third normal forms} : (\chi_0 \chi_1^2 \chi_2 \pm \chi_1^k \chi_3) \text{ near } C_{Q_0} \cap C_{R_k}. \quad (42)$$

We end this section with two lemmas on asymptotics of oscillatory integrals.

Lemma 32 *There is a sequence $(c_j)_j \in \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R})$ such that for $k > 1$*

$$\int_0^\infty \left(\int_{\mathbb{R}} e^{i\lambda t r^k} a(t, r) dt \right) dr \sim \sum_{j=0}^\infty \lambda^{-\frac{j+1}{k}} c_j(a), \quad (43)$$

where :

$$c_l = \frac{(-1)^l}{k} \frac{1}{l!} (\Lambda_l(t) \otimes \delta_0^{(l)}(r)), \quad (44)$$

with $\Lambda_l = \mathcal{F}(x_-^{\frac{l+1-k}{k}})$ and $x_- = \max(-x, 0)$.

Proof. We define $\hat{g}(\tau, r) = \mathcal{F}_t(a(t, r))(\tau)$, where \mathcal{F}_t is the partial Fourier transform with respect to t . Then we obtain

$$\int_0^\infty \left(\int_{\mathbb{R}} e^{i\lambda t r^k} a(t, r) dt \right) dr = \int_0^\infty \hat{g}(-\lambda r^k, r) dr = \lambda^{-\frac{1}{k}} \int_0^\infty \hat{g}(-r^k, \frac{r}{\lambda^{\frac{1}{k}}}) dr. \quad (45)$$

Taking the Taylor series in r of $\hat{g}(\tau, r)$, at the origin, gives

$$\hat{g}(-r^k, \frac{r}{\lambda^{\frac{1}{k}}}) = \sum_{l=0}^N \frac{\lambda^{-\frac{l}{k}}}{l!} r^l \frac{\partial^l \hat{g}}{\partial r^l}(-r^k, 0) + \lambda^{-\frac{N+1}{k}} R(r, \lambda),$$

Straightforward computations shows that

$$\lambda^{-\frac{1}{k}} \int_0^\infty \hat{g}(-r^k, \frac{r}{\lambda^{\frac{1}{k}}}) dr = \frac{1}{k} \sum_{l=0}^N \frac{\lambda^{-(\frac{l+1}{k})}}{l!} \int_{-\infty}^0 \frac{\partial^l \hat{g}}{\partial r^l}(r, 0) |r|^{\frac{l+1-k}{k}} dr + \mathcal{O}(\lambda^{-\frac{N+1}{k}}).$$

With $x_- = \max(-x, 0)$ and $\Lambda_l(r) = \mathcal{F}(x_-^{\frac{l+1-k}{k}})(r)$ the lemma follows. ■

Lemma 33 *There is a sequence $(c_j)_j \in \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R})$ such that for $k > 1$*

$$\int_0^\infty e^{i\lambda r^k} a(r) dr \sim \sum_{j=0}^\infty \lambda^{-\frac{j+1}{k}} c_j(a), \quad (46)$$

with :

$$c_j(a) = \frac{(-1)^j}{k} \Gamma\left(\frac{j}{k}\right) \exp(i\pi \frac{j}{2k}) \frac{\partial^j a}{\partial r^j}(0). \quad (47)$$

Proof. We use the Bernstein-Sato polynomial, see e.g. [16]. We write

$$\int_0^\infty e^{i\lambda r^k} a(r) dr = \frac{1}{2i\pi} \int_\gamma e^{i\pi \frac{z}{2}} \Gamma(z) \lambda^{-z} \left(\int_0^\infty a(r) r^{-kz} dr \right) dz,$$

with $\gamma =]c - i\infty, c + i\infty[$, $\operatorname{Re}(c) < k^{-1}$. Since for all positive r we have

$$\frac{\partial^k}{\partial r^k} (r^k)^{1-z} = r^{-kz} \prod_{j=1}^k (j - kz),$$

we can compute the asymptotic by the residue method. All poles are simple and, by pushing of the complex path of integration to the right, we obtain

$$\lim_{z \rightarrow \frac{l}{k}} \frac{(z - \frac{l}{k})}{\prod_{j=1}^k (j - kz)} e^{i\pi \frac{z}{2}} \Gamma(z) \lambda^{-z} (-1)^k \int_0^\infty \frac{\partial^k a}{\partial r^k}(r) r^{k-kz} dr = \mu_l \frac{\partial^{l-1} a}{\partial r^{l-1}}(0).$$

Straightforward computations then show that $\mu_l = (-1)^k \Gamma(\frac{l}{k}) \exp(i\pi \frac{l}{2k})$. ■

6 Proofs of the main theorems

We start with the simpler case of T being a total period of $d\Phi_t(z_0)$. Afterwards we study the contributions of non-total periods T , distinguishing out particular case of a function p_2 whose restriction to the linear subspace \mathfrak{F}_T has constant sign. In the following we suppose, without loss of generality, that the support of the amplitude contains only one non-zero period of the linearized flow.

6.1 Blow-up and partition of the sphere

We will apply the results of sections 4 and 5, and we recall that

$$R(x, \xi) = S(T, x, \xi) - \langle x, \xi \rangle,$$

$$(t - T)g(t, x, \xi) + tE_c = (t - T)(Q_t(x, \xi) + h(t, x, \xi)).$$

By a time translation and with the polar coordinates $(x, \xi) = r\theta$, $\theta \in \mathbb{S}^{2n-1}(\mathbb{R})$, we obtain for the top order part of $(2\pi h)^n \gamma_2(E_c, h)$:

$$I(T, h) = \int_{\mathbb{R} \times [0, \infty[} \int_{\mathbb{S}^{2n-1}} a(t, r\theta) e^{\frac{i}{h} \Psi(t, r, \theta)} r^{2n-1} dt dr d\theta,$$

where $d\theta$ is the standard surface measure on the sphere.

Partition of unity on the sphere.

Since $C_{Q_T} \cap \mathbb{S}^{2n-1}(\mathbb{R})$ is compact, we can introduce a finite partition of unity

$$\sum_{i \in I} \Omega_i^1(\theta) + \sum_{j \in J} \Omega_j^2(\theta) + \sum_{l \in L} \Omega_l^3(\theta) = 0 \text{ on } \mathbb{S}^{2n-1}(\mathbb{R}),$$

with the property that $Q_T(\theta) \neq 0$ on $\bigcup \text{supp}(\Omega_i^1)$, $R_k(\theta) \neq 0$ on $\bigcup \text{supp}(\Omega_j^2)$ and $C_{Q_T} \cap C_{R_k} \subset \bigcup \text{supp}(\Omega_l^3)$. We split up the integral $I(T, h)$ according to this partition of unity and use the normal forms of Theorem 31. On any chart let be J_χ the Jacobian of the relevant diffeomorphism of blow-up from Theorem 31. For j in I , J and L respectively, we define

$$a_{i,j}(\chi_0, \chi_1, \dots, \chi_{2n}) = (\chi^{-1})^*(\Omega_j^i(\theta)a(t, r\theta)r^{2n-1}|J_\chi|^{-1}), \quad i \in \{1, 2, 3\}, \quad (48)$$

Then, from Theorem 31, we obtain the local contributions

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^{2n-1}} \Omega_i^1(\theta)a(t, r\theta)e^{\frac{i}{h}\Psi(t, r\theta)}r^{2n-1}dt dr d\theta \\ &= \int_{\mathbb{R} \times [0, \infty[} e^{\pm \frac{i}{h}\chi_0\chi_1^2} A_{1,i}(\chi_0, \chi_1) d\chi_0 d\chi_1, \end{aligned}$$

for the first normal forms. Also

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^{2n-1}} \Omega_j^2(\theta)a(t, r\theta)e^{\frac{i}{h}\Psi(t, r\theta)}r^{2n-1}dt dr d\theta \\ &= \int_{\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}} A_{2,j}(\chi_0, \chi_1, \chi_2)e^{\frac{i}{h}(\chi_0\chi_1^2\chi_2 \pm \chi_1^k)} d\chi_0 d\chi_1 d\chi_2, \end{aligned}$$

for the second normal forms and for the third normal forms

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^{2n-1}} \Omega_l^3(\theta)a(t, r\theta)e^{\frac{i}{h}\Psi(t, r\theta)}r^{2n-1}dt dr d\theta \\ &= \int_{\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^2} A_{3,l}(\chi_0, \dots, \chi_3)e^{\frac{i}{h}(\chi_0\chi_1^2\chi_2 \pm \chi_1^k\chi_3)} d\chi_0 \dots d\chi_3, \end{aligned}$$

where the amplitudes are respectively given by

$$A_{1,i}(\chi_0, \chi_1) = \int a_{2,i}(\chi_0, \chi_1, \dots, \chi_{2n}) d\chi_2 \dots d\chi_{2n}, \quad (49)$$

$$A_{2,j}(\chi_0, \chi_1, \chi_2) = \int a_{2,j}(\chi_0, \chi_1, \dots, \chi_{2n}) d\chi_3 \dots d\chi_{2n}, \quad (50)$$

$$A_{3,l}(\chi_0, \chi_1, \chi_2, \chi_3) = \int a_{3,l}(\chi_0, \chi_1, \dots, \chi_{2n}) d\chi_4 \dots d\chi_{2n}. \quad (51)$$

It is convenient, for the calculations below, to introduce $A_{j,i} = \chi_1^{2n-1} \tilde{A}_{j,i}$, for $j \in \{1, 2, 3\}$, c.f. Remark [30]. By construction the functions $A_{j,i}$ are of compact support in their system of coordinates.

Remark 34 We obtain for each new phases the following critical sets

$$\begin{cases} \mathfrak{C}(\chi_0\chi_1^2) = \{\chi_1 = 0\}, \\ \mathfrak{C}(\chi_0\chi_1^2\chi_2 + \chi_1^k) = \{\chi_1 = 0\}, \\ \mathfrak{C}(\chi_0\chi_1^2\chi_2 + \chi_1^k\chi_3) = \{\chi_1 = 0\}. \end{cases}$$

Where $\mathfrak{C}(f)$ denotes the critical set of a function f .

6.2 Analysis in the case of a total period

First normal forms.

We note \mathcal{F} and \mathcal{F}_t the total and partial Fourier transform with respect to t . For an amplitude of the form $a(r, t) = \mathcal{O}(r^{2n-1})$ and $k = 2$, where $r = \chi_1$, Lemma 32 shows that the first non-zero coefficient is obtained for $l_0 = 2n - 1$. With $\lambda = h^{-1}$, this gives the contribution

$$\frac{\lambda^{-n}}{(2n-1)!} c_{2n-1}(A_{1,i}) = \lambda^{-n} \langle \Lambda_{2n-1}(t), \tilde{A}_{1,i}(t, 0) \rangle,$$

where $\Lambda_{2n-1}(t) = \mathcal{F}(x_-^{n-1})(t)$. If we define the distribution

$$\Lambda_{1,i}(a) = \langle \Lambda_{2n-1}(\chi_0), \int (\chi^{-1})^*(\Omega_i^1(\theta)a(t, r\theta)|J\chi|^{-1})(\chi_0, 0, \chi_2, \dots, \chi_{2n})d\chi_2\dots d\chi_{2n} \rangle,$$

we obtain, as $\lambda \rightarrow +\infty$, the asymptotic equivalent

$$\int_{\mathbb{R}} \int_0^\infty e^{i\lambda\chi_0\chi_1^2} A_{1,i}(\chi_0, \chi_1) d\chi_0 d\chi_1 = -\frac{1}{2} \Lambda_{1,i}(a) \lambda^{-n} + \mathcal{O}(\lambda^{-n-\frac{1}{2}}). \quad (52)$$

Second normal forms.

If we write a for $\tilde{A}_{2,j}$, $\lambda = h^{-1}$ and (t, r, v) for (χ_0, χ_1, χ_2) , then we have to analyze the asymptotic behavior of the oscillatory integral

$$I_2(\lambda) = \int_0^\infty \int_{\mathbb{R}^2} e^{i\lambda(tr^2v+r^k)} a(t, r, v) dt dv r^{2n-1} dr \quad (53)$$

If we let $\hat{a}(\tau, r, v) = \mathcal{F}_t(a(t, r, v))(\tau)$, the Fourier transform with respect to t , then we find by easy manipulations, that

$$I_2(\lambda) = \lambda^{-1} \int_0^\infty r^{2n-3} e^{i\lambda r^k} \int_{\mathbb{R}} \hat{a}(-w, r, \frac{w}{\lambda r^2}) dw dr, \quad (54)$$

where we made the change of variables $(v, r) \rightarrow (w, r)$, $w = \lambda r^2 v$. We now Taylor expand up till order $n - 3$:

$$\hat{a}(-w, r, \frac{w}{\lambda r^2}) = \sum_{j=0}^{n-3} c_j(-w, r) (\frac{w}{\lambda r^2})^j + R_{n-2}(r, w, \lambda),$$

where

$$c_j(w, r) = \frac{1}{j!} \frac{\partial^j \hat{a}}{\partial v^j}(w, r, 0),$$

and

$$R_{n-2}(r, w, \lambda) = (\frac{w}{\lambda r^2})^{n-2} \int_0^1 \partial_r^{n-2} \hat{a}(-w, r, \frac{tw}{\lambda r^2}) \frac{(1-t)^{n-3}}{(n-3)!} dt.$$

Substitution in Eq.(54) leads to

$$I_2(\lambda) = \sum_{j=0}^{n-3} J_j(\lambda) + r_{n-2}(\lambda),$$

where

$$J_j(\lambda) = \lambda^{-(1+j)} \int_0^\infty e^{i\lambda r^k} a_j(r) r^{2n-3-2j} dr,$$

$$a_j(r) = \int_{\mathbb{R}} c_j(-w, r) w^j dw = \frac{(-i)^j}{j!} ((\frac{\partial^2}{\partial v \partial t})^j a)(0, r, 0),$$

and

$$r_{n-2}(\lambda) = \lambda^{-1} \int_0^\infty \int_{\mathbb{R}} R_{n-2}(r, w, \lambda) e^{i\lambda r^k} r^{2n-3} dr dw.$$

Observe that the oscillatory integrals $J_j(\lambda)$ can be treated individually using Lemma 33, but we first analyze the remainder term, $r_{n-2}(\lambda)$, which equals

$$\frac{1}{\lambda^{n-1}} \int_0^\infty r e^{i\lambda r^k} \int_{\mathbb{R}} \int_0^1 w^{n-2} \partial_v^{n-2} \hat{a}(-w, r, \frac{tw}{\lambda r^2}) \frac{(1-t)^{n-3}}{(n-3)!} dt dw dr. \quad (55)$$

We split the integral with respect to dr as :

$$\int_0^\infty dr = \int_0^A dr + \int_A^\infty dr,$$

where $A = A(\lambda)$ will be chosen below. We accordingly split $r_{n-2}(\lambda)$ as $r_{n-2}^{1,A}(\lambda) + r_{n-2}^{2,A}(\lambda)$ and $r_{n-2}^{1,A}(\lambda)$ is given by Eq.(55), with the integral restrained to $[0, A]$. Easy estimates then show that $r_{n-2}^{1,A}(\lambda) \leq C A^2 \lambda^{-(n-1)}$ with C independent of λ .

Next, for $r_{n-2}^{2,A}(\lambda)$ we do an integration by part with respect to t , this leads to

$$\begin{aligned} r_{n-2}^{2,A}(\lambda) &= \lambda^{-(n-1)} \int_A^\infty r e^{i\lambda r^k} \int_{\mathbb{R}} \frac{w^{n-2}}{(n-2)!} \partial_v^{n-2} \hat{a}(-w, r, 0) dw dr \\ &+ \lambda^{-n} \int_A^\infty \frac{e^{i\lambda r^k}}{r} \int_{\mathbb{R}} \int_0^1 w^{n-1} \partial_v^{n-1} \hat{a}(-w, r, \frac{tw}{\lambda r^2}) \frac{(1-t)^{n-2}}{(n-2)!} dt dw dr. \end{aligned} \quad (56)$$

We then observe that the first integral in Eq.(56) is equal to

$$\begin{aligned} \lambda^{-(n-1)} \int_0^\infty r e^{i\lambda r^k} \int_{\mathbb{R}} \frac{w^{n-2}}{(n-2)!} \partial_v^{n-2} \hat{a}(-w, r, 0) dw dr &+ \mathcal{O}(A^2 \lambda^{-(n-1)}) \\ &= J_{n-2}(\lambda) + \mathcal{O}(A^2 \lambda^{-(n-1)}), \end{aligned}$$

by similar estimates as for $r_{n-2}^{1,A}(\lambda)$. Finally, the last integral in Eq.(56) can be estimated by

$$\frac{\lambda^{-n}}{(n-2)!} \int_A^\infty \int_{\mathbb{R}} \frac{|w|^{n-1}}{r} \|\partial_v^{n-1} \hat{a}(-w, r, \bullet)\|_\infty dw dr \leq C \lambda^{-n} |\log(A)|,$$

remembering that \hat{a} has a compact support in v . In conclusion, we find that

$$I_2(\lambda) = \sum_{j=0}^{n-2} J_j(\lambda) + \mathcal{O}(\lambda^{-(n-1)} A^2) + \mathcal{O}(\lambda^{-n} \log(A)) = \sum_{j=0}^{n-2} J_j(\lambda) + \mathcal{O}(\lambda^{-n} \log(\lambda)),$$

where we have chosen $A = \lambda^{-\frac{1}{2}}$.

Now by Lemma 32, and since $a_0(r) = \tilde{A}_{2,j}(0, r, 0)$, we obtain

$$J_0(\lambda) = \mu_0 \lambda^{\frac{2-k-2n}{k}} \frac{\partial^{2n-3} a_0}{\partial r^{2n-3}}(0) + \mathcal{O}(\lambda^{\frac{1-2n-k}{k}}), \quad (57)$$

with $\mu_0 = -\frac{1}{k} \Gamma(\frac{n-2}{k}) \exp(i\pi \frac{n-1}{k})$. While, if $j > 0$, the same lemma shows that

$$J_j(\lambda) = \mathcal{O}(\lambda^{-\frac{2n+(j+1)(k-2)}{k}}), \quad 0 < j \leq n-2.$$

All of the latter are dominated by the top order term of $J_0(\lambda)$, and dominate the remainder, which is $\mathcal{O}(\lambda^{-n} \log(\lambda))$, since for $j = n-2$ and for all $k \geq 3$:

$$\frac{2n + (k-2)(j+1)}{k} = n-1 + \frac{2}{k} < n.$$

As concerns the error term in (56), this also dominates the remainder term, since $n > \frac{k-2}{k-1}$ for all $n \geq 2$. Since $h = \lambda^{-1}$, we have shown that the contribution of the second normal form to the asymptotics of $I(T, h)$ is :

$$I_2(h) = \mu_0 \tilde{A}_{2,j}(0, 0, 0) h^{\frac{2n+k-2}{k}} + \mathcal{O}(h^{\frac{2n+k-1}{k}}). \quad (58)$$

Third normal forms.

We use the same strategy as for second normal forms. If $\lambda = h^{-1}$, $(t, r, v, s) = (\chi_0, \chi_1, \chi_2, \chi_3)$ and $a = \tilde{A}_{3,j}$, we write

$$I_3(\lambda) = \int_0^\infty \int_{\mathbb{R}^3} e^{i\lambda(tr^2v+r^ks)} a(t, r, v, s) dt dv ds r^{2n-1} dr,$$

with $\hat{a}(\tau, r, v, s) = \mathcal{F}_t(a(t, r, v, s))(\tau)$, a Taylor expansion, up till order $n-3$, gives again

$$\hat{a}(-w, r, \frac{w}{\lambda r^2}, s) = \sum_{j=0}^{n-3} c_j(-w, r, s) \left(\frac{w}{\lambda r^2}\right)^j + R_{n-2}(r, w, \lambda, s),$$

with

$$\begin{aligned} R_{n-2}(r, w, \lambda, s) &= \left(\frac{w}{\lambda r^2}\right)^{n-2} \int_0^1 \partial_r^{n-2} \hat{a}(-w, r, \frac{tw}{\lambda r^2}, s) \frac{(1-t)^{n-3}}{(n-3)!} dt, \\ c_j(w, r, s) &= \frac{1}{j!} \frac{\partial^j \hat{a}}{\partial v^j}(w, r, 0, s), \\ a_j(r, s) &= \int_{\mathbb{R}} c_j(-w, r, s) w^j dw = (-i)^j \left(\frac{\partial^2}{\partial v \partial t}\right)^j a(0, r, 0, s). \end{aligned}$$

This leads to

$$\begin{aligned} I_3(\lambda) &= \sum_{j=0}^{n-3} K_j(\lambda) + r_{n-2}(\lambda) \\ &= \sum_{j=0}^{n-3} \lambda^{-(1+j)} \int e^{i\lambda r^k s} a_j(r, s) r^{2n-3-2j} dr ds + r_{n-2}(\lambda). \end{aligned}$$

The $K_j(\lambda)$ can be treated individually via Lemma 32. For $j=0$, we obtain

$$\begin{aligned} K_0(\lambda) &= \lambda^{-1} \int_0^\infty \int_{\mathbb{R}} e^{i\lambda r^k s} a_0(r, s) r^{2n-3} dr ds \\ &\sim \lambda^{-1} \left(\sum_{l=0}^N \lambda^{-\frac{l+1}{k}} c_l(a_0(r, s) r^{2n-3}) + \mathcal{O}(\lambda^{-\frac{N+2}{k}}) \right). \end{aligned}$$

The leading term, obtained for $l = 2n-3$, is

$$\lambda^{-(\frac{2n-2+k}{k})} c_{2n-3}(a_0(r, s) r^{2n-3}) = -\frac{1}{k} \lambda^{-(\frac{2n-2+k}{k})} \left\langle \mathcal{F}(x_-^{\frac{2n-2-k}{k}})(s), a_0(0, s) \right\rangle.$$

Hence for our amplitude we have the main contribution

$$-\frac{1}{k} \lambda^{-(\frac{2n-2+k}{k})} \left\langle \mathcal{F}(x_-^{\frac{2n-2-k}{k}})(\chi_3), \tilde{A}_{3,l}(0, 0, 0, \chi_3) \right\rangle + \mathcal{O}(\lambda^{-(\frac{2n-1+k}{k})}). \quad (59)$$

Like for the second normal forms, the other terms $K_j(\lambda)$, with $j > 0$, and the remainder $r_{n-2}(\lambda)$ give contributions of strictly lower orders.

Finally, on each local chart the main contributions are

$$\begin{cases} \text{first normal forms} & : c_{0,1}(a)\lambda^{-n}, \\ \text{second normal forms} & : c_{0,2}(a)\lambda^{\frac{2-2n-k}{k}}, \\ \text{third normal forms} & : c_{0,3}(a)\lambda^{\frac{2-2n-k}{k}}. \end{cases}$$

The contributions of charts 2 and 3 are dominant, since $\frac{2-2n-k}{k} > -n$, $\forall k > 2$.

Remark 35 The proofs above show in fact much more than just an asymptotic equivalent for $I(T, h)$, and therefore for $\gamma_2(E_c, h)$. They show the existence of a limited asymptotic expansion in the case of indefinite Q_T :

$$I(T, h) = \sum c_\nu h^{\frac{2n+k-2+\nu}{k}} + \mathcal{O}(h^n |\log(h)|), \quad (60)$$

where the sum is over all ν such that $\frac{2n+k-2+\nu}{k} < n$, or $\nu < (k-2)(n-1)$, and of a complete asymptotic expansion if Q_T is definite. A similar remark applies for the case of a non-total period, which we examine in the next section.

We now compute the leading term of the expansion in case of a non-definite Q_T and for T a total period of the linearized flow.

Case of an empty intersection of cones.

With $R_k \neq 0$ on $C_{p_2} \setminus \{0\}$, we can assume that R_k is positive on $C_{p_2} \setminus \{0\}$. The main contribution is here given by the second normal form and is

$$\begin{aligned} & \int_{\mathbb{R} \times [0, \infty[\times \mathbb{R}} A_{2,j}(\chi_0, \chi_1, \chi_2) e^{i\lambda(\chi_0 \chi_1^2 \chi_2 + \chi_1^k)} d\chi_0 d\chi_1 d\chi_2 \\ & = \mu_k \lambda^{\frac{2-k-2n}{k}} \tilde{A}_{2,j}(0, 0, 0) + \mathcal{O}(\lambda^{\frac{2-k-2n}{k} - \frac{1}{k}}), \end{aligned}$$

with μ_k given by Lemma 33. By definition of χ , the amplitude $\tilde{A}_{2,j}(0, 0, 0)$ is

$$\int (\chi^{-1})^*(\Omega_j^2(\theta) |R_k(\theta) + r\tilde{R}(r, \theta)|^{-\frac{(2n-1)}{k}} a(t+T, r\theta) |J\chi|^{-1})(0, \chi_3, \dots, \chi_{2n}) d\chi_3 \dots d\chi_{2n}.$$

For $z \in \mathbb{R}^3$, we write this delta-Dirac distribution as an oscillatory integral

$$\frac{1}{(2\pi)^3} \int e^{-i\langle z, (\chi_0, \chi_1, \chi_2)(t, r, \theta) \rangle} \Omega_j^2(\theta) |R_k(\theta) + r\tilde{R}(r, \theta)|^{-\frac{2n-1}{k}} a(t+T, r\theta) dz dt dr d\theta.$$

If we use $y_2 = z_2 |R_k(\theta) + r\tilde{R}(r, \theta)|^{\frac{1}{k}}$, integration in (y_2, r) gives

$$(2\pi)^2 \tilde{A}_{2,j}(0, 0, 0) = \int e^{-i\langle (z_1, z_3), (\chi_0, \chi_2)(t, 0, \theta) \rangle} \Omega_j^2(\theta) |R_k(\theta)|^{-\frac{2n}{k}} a(t+T, 0) dt d\theta dz_1 dz_3.$$

Since $\chi_0(t, 0, \theta) = t|R_k(\theta)|^{-\frac{2}{k}}$, with $y_1 = z_1|R_k(\theta)|^{-\frac{2}{k}}$, by integration in (t, y_1)

$$(2\pi)\tilde{A}_{2,j}(0, 0, 0) = a(T, 0) \int e^{-iz_3 p_2(\theta)} \Omega_j^2(\theta) |R_k(\theta)|^{-\frac{2n-2}{k}} d\theta dz_3.$$

We define the Liouville measure dL_{p_2} on $C_{p_2} \cap \mathbb{S}^{2n-1}$ via $dL_{p_2}(\theta) \wedge dp_2(\theta) = d\theta$. Since charts associated to second normal forms cover the trace of the cone, by summation over the partition of unity we obtain

$$I(T, h) = h^{\frac{2n+k-2}{k}} (\mu_k a(T, 0) \int_{C_{p_2} \cap \mathbb{S}^{2n-1}} |R_k(\theta)|^{-\frac{2n-2}{k}} dL_{p_2}(\theta) + \mathcal{O}(h^{\frac{1}{k}})),$$

where θ are now local coordinates on the surface $C_{p_2} \cap \mathbb{S}^{2n-1}$. The top-order contribution to the trace formula follows from $\gamma_2(E_c, T, h) = (2\pi)^{-n-1} h^{-n} I(T, h)$.

Case of a non-empty intersection of cones.

Here the main contribution is given by the normal forms 2 and 3. The local contribution of any chart associated to second normal forms can be computed like in the previous section. The contribution of the third normal forms is given by equation (59) with the amplitude $\tilde{A}_{3,l}(0, 0, 0, \chi_3)$. Let be $z = (z_1, z_2, z_3)$, we use again an oscillatory representation of the delta-Dirac distribution via

$$\frac{1}{(2\pi)^3} \int e^{-i\langle z, (\chi_0, \chi_1, \chi_2) \rangle} \left\langle \mathcal{F}(x_-^{\frac{2n-2-k}{k}})(\chi_3), \tilde{A}_{3,l}(\chi_0, \dots, \chi_3) \right\rangle d\chi_0 d\chi_1 d\chi_2 dz.$$

Since $(\chi_0, \chi_1) = (t, r)$, integration w.r.t. $(z_1, z_3, \chi_0, \chi_1)$ gives

$$\begin{aligned} & 2\pi \left\langle \mathcal{F}(x_-^{\frac{2n-2-k}{k}})(\chi_3), \tilde{A}_{3,l}(0, 0, 0, \chi_3) \right\rangle \\ &= \int e^{-iz_2 \chi_2(0, 0, \theta)} \left\langle \mathcal{F}(x_-^{\frac{2n-2-k}{k}})(\chi_3(0, 0, \theta)), \Omega_l^3(\theta) a(T, 0) \right\rangle d\theta dz_2. \end{aligned}$$

A classical result, see [8] volume 1 page 167, is

$$\mathcal{F}(x_-^{\frac{2n-2-k}{k}})(\chi_3) = \Gamma\left(\frac{2n-2}{k}\right) \exp(i\pi \frac{n-1}{k}) (\chi_3 + i0)^{-\frac{2n-2}{k}}. \quad (61)$$

By construction $(\chi_2, \chi_3)(0, 0, \theta) = (p_2(\theta), R_k(\theta))$ and with the notations of Theorem 4 we obtain, using again p_2 as a local coordinate

$$\mu_k a(T, 0) \int_{C_{p_2} \cap \mathbb{S}^{2n-1}} (R_k(\theta) + i0)^{-\frac{2n-2}{k}} \Omega_l^3(\theta) dL_{p_2}(\theta).$$

But $(R_k(\theta) + i0)^{-\frac{2n-2}{k}} = R_k(\theta)^{-\frac{2n-2}{k}}$ on a chart associated to the second normal form and the total contribution arising from normal forms 2 and 3 is

$$I(T, h) = h^{\frac{2n-2+k}{k}} (\mu_k a(T, 0) \int_{C_{p_2} \cap \mathbb{S}^{2n-1}} (R_k(\theta) + i0)^{-\frac{2n-2}{k}} dL_{p_2}(\theta) + \mathcal{O}(h^{\frac{1}{k}})). \quad (62)$$

Since $a(T, 0) = \hat{\varphi}(T) \exp(iTp^1(z_0))$, see formula (66) below, this proves Theorem 4 for a total period. \blacksquare

6.3 Case of a non-total period

Let be T a non total period of the linearized flow. We can assume, up to a permutation of coordinates, that $\mathfrak{F}_T = \{z = (z_1, z_2) \in \mathbb{R}^{l_T} \times \mathbb{R}^{2n-l_T} / z_2 = 0\}$. We can apply the Morse lemma with parameter since the phase function at time T is only degenerate in z along \mathfrak{F}_T (cf. Corollary 10). The quadratic part $S_2(t, x, \xi)$ of the function S is given by

$$\Phi_t(\partial_\xi S, \xi) = d\Phi_t(0)(\partial_\xi S_2, \xi) + \mathcal{O}(\|(x, \xi)\|^2) = (x, \partial_x S_2) + \mathcal{O}(\|(x, \xi)\|^2).$$

Using Theorem 5.3 of [9] we can assume that $\det[\frac{\partial^2 S}{\partial x \partial \eta}] \neq 0$ and the function S_2 is well determined locally. Then we have the following facts

$$\begin{cases} d_z \Psi(t, z) = 0 \Leftrightarrow z = z_0, \forall t, \\ \text{Hess}_{z_2}(\Psi)(T, z_0) = \partial_{z_2}^2 \Psi(T, z_0), \text{ is invertible.} \end{cases}$$

By the Morse lemma, after a change of variable $z \rightarrow \tilde{z}$ and calling \tilde{z} again z , we have

$$\Psi(t, z) = q(z_2) + \Psi(t, z_1, z_2(t, z_1)) = q(z_2) + \tilde{\Psi}(t, z_1),$$

and by Corollary 10 again, $q = \frac{1}{2}q_T = p_2|_{\mathfrak{F}_T^\perp}$. In the following, we note

$$\begin{aligned} R(t, z_1) &\equiv R(z_1, z_2(t, z_1)), \\ g(t, z_1) &\equiv g(t, z_1, z_2(t, z_1)), \\ \Psi(t, z_1) &\equiv \Psi(t, z_1, z_2(t, z_1)) = R(z_1, z_2(t, z_1)) + (t - T)g(t, z_1, z_2(t, z_1)). \end{aligned}$$

With these conventions we can write

$$I(T, h) = \int_{z_1 \in \mathbb{R}^{l_T}} \int_{z_2 \in \mathbb{R}^{2n-l_T}} e^{\frac{i}{2h}q_T(z_2)} \tilde{a}(t, z_1, z_2) e^{\frac{i}{h}(R(t, z_1) + (t-T)g(t, z_1))} dt dz_1 dz_2,$$

where \tilde{a} is the new amplitude after change of variables due to the Morse lemma. The stationary phase method applied to the z_2 -integral gives

$$\int_{z_2 \in \mathbb{R}^{2n-l_T}} e^{\frac{i}{2h}q_T(z_2)} \tilde{a}(t, z_1, z_2) dz_2 = \sum_{\nu=0}^N c_\nu h^{n+\nu-d_T} A_\nu(t, z_1) + \mathcal{O}(h^{N+\nu-d_T+1}), \quad (63)$$

and in particular for the leading term we have

$$A_0(t, z_1) = a(t, z_1, 0) \left| \frac{\partial z_2(t, z_1)}{\partial(z_1, t)} \right|^{-1},$$

$$c_0 = \frac{(2\pi)^{n-d_T} \exp(i\frac{\pi}{4} \text{sgn}(q_T))}{|\det(q_T)|^{\frac{1}{2}}}.$$

We now distinguish two different cases : the quadratic form Q_T , that is p_2 restricted to \mathfrak{F}_T , is definite or non-definite. In the first case only the normal forms $\pm \chi_0 \chi_1^2$ will occur, i.e. the knowledge of $d\Phi_t(z_0)$ is sufficient. In the second case the main contribution involves R_k , and hence the operator $d^{k-1}\Phi_t(z_0)$.

The case Q_T definite.

Up to a complex conjugation we can assume that Q_T is positive. With polar coordinates $z_1 = (r\theta)$ and $a|_{z_2=0} = a(t, r\theta, 0)$, the new amplitude is

$$A_0(\chi_0, \chi_1) = \int (\chi^{-1})^* (a(t, r\theta, 0) r^{l_T-1} |J\chi(t, r, \theta)|) d\chi_2 \dots d\chi_{2d_T}, \quad (64)$$

with $A_0(\chi_0, \chi_1) = \chi_1^{l_T-1} \tilde{A}_0(\chi_0, \chi_1)$. Lemma 32 shows that

$$\int_{\mathbb{R}} \int_0^\infty A_0(\chi_0, \chi_1) e^{\frac{i}{h} \chi_0 \chi_1^2} d\chi_0 d\chi_1 = -\frac{1}{2} h^{d_T} \left\langle \mathcal{F}(x_-^{d_T-1})(\chi_0), \tilde{A}_0(\chi_0, 0) \right\rangle + \mathcal{O}(h^{d_T+\frac{1}{2}}).$$

With $\Lambda(\chi_0) = \mathcal{F}(x_-^{d_T-1})(\chi_0)$, substituting the definition of \tilde{A}_0 , gives

$$\left\langle \Lambda(\chi_0), \tilde{A}_0(\chi_0, 0) \right\rangle = 2\pi \int e^{izr} \Lambda(\chi_0(t, r, \theta)) a(t, r\theta, 0) dt dr d\theta dz.$$

But by construction $\chi_1 = r$, and since we have localized the amplitude near T

$$\left\langle \Lambda(\chi_0), \tilde{A}_0(\chi_0, 0) \right\rangle = -e^{i\frac{\pi}{2}(d_T-1)} \Gamma(d_T) \int \left\langle (tQ_t(\theta) - i0)^{-d_T}, a(t+T, 0) \right\rangle d\theta dt. \quad (65)$$

We can now use a result of [6], also used in [10] and [2]. Since the propagator $\text{Exp}(\frac{i}{h}tP_h)$ is a FIO associated to the Lagrangian manifold of the flow,

$$\Lambda = \{(t, \tau, x, \xi, y, \eta) / (x, \xi) = \Phi_t(y, \eta), \tau = -p(x, \xi)\},$$

it's principal symbol in the coordinates (t, y, η) is given by the half-density

$$\exp(i \int_0^t p^1(\Phi_s(y, \eta)) ds) |dt dy d\eta|^{\frac{1}{2}}.$$

The representation of the propagator with the kernel

$$\frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(t,x,\eta) - \langle y, \eta \rangle)} (\alpha(t, x, \eta) + h\alpha_1(t, x, \eta, h)) dy,$$

leads to the half-density

$$\alpha(t, x, \eta) |dtdxd\eta|^{\frac{1}{2}} = \alpha(t, x, \eta) \frac{|dtdy d\eta|^{\frac{1}{2}}}{|\det(S''_{x,\eta})|^{\frac{1}{2}}}.$$

For our unique critical point z_0 we obtain

$$\alpha(t, z_0) = |\det(S''_{x,\eta})(t, z_0)|^{\frac{1}{2}} \exp(i \int_0^t p^1(\Phi_s(z_0)) ds) = \exp(itp^1(z_0)). \quad (66)$$

If there is no period of $d\Phi_t(z_0)$ on $\text{supp}(\hat{\varphi})$ Theorem 5.6 of [6] gives

$$\text{Tr}(\psi^w(x, hD_x)\varphi(\frac{P_h - E_c}{h})\theta(P_h)) \simeq \frac{1}{2\pi} e^{i\frac{\pi}{2}m_0} \int_{\mathbb{R}} \frac{\hat{\varphi}(t) \exp(itp^1(z_0))}{|\det(\text{Id} - d\Phi_t(z_0))|^{\frac{1}{2}}} dt,$$

for a certain m_0 . The denominator has a zero of order d_T in $t = T$, hence for all $t \neq T$ in a sufficiently small neighborhood of T , we have

$$a(t, 0) = \frac{(t - T)^{d_T}}{|\det(\text{Id} - d\Phi_t(z_0))|^{\frac{1}{2}}} \hat{\varphi}(t) \exp(itp^1(z_0)),$$

Finally, since the contribution is smooth in t , we obtain that

$$K(T) = -\frac{\Gamma(d_T)}{2} \frac{\exp(i\frac{\pi}{4}\text{sgn}(q_T))}{(2\pi)^{1+d_T} |\det(q_T)|^{\frac{1}{2}}} \exp(i\pi \frac{d_T - 1}{2} \text{sign}(Q_T)),$$

$$\Lambda(\varphi) = \left\langle (t - T - i0)^{-d_T}, \frac{(t - T)^{d_T} \exp(itp^1(z_0))}{|\det(\text{Id} - d\Phi_t(z_0))|^{\frac{1}{2}}} \hat{\varphi}(t) \right\rangle,$$

and this completes the proof of Theorem 3.

Remark 36 If there are no rational relations between the eigenvalues of Q^+ and Q^- all contributions of the non-zero periods of $d\Phi_t(z_0)$ are given by Theorem 3, as is shown by Proposition 15. This gives a total contribution

$$\sum_{T \in \text{supp}(\hat{\varphi}) \setminus \{0\}} \gamma(E_c, T, h) = \sum_{T \in \text{supp}(\hat{\varphi}) \setminus \{0\}} (K(T) \Lambda_T(\varphi) + \mathcal{O}(h^{\frac{1}{2}})),$$

where the summation is over the non-zero periods of $d\Phi_t(z_0)$.

The case Q_T indefinite.

Here we must study carefully the solutions of the implicit function theorem. If z_0 is zero then, by uniqueness, we have $z_2(t, 0) = 0$ for all t , and

$$\begin{cases} d_{z_2} \Psi(t, z_1, z_2(t, z_1)) = 0, \\ \det(\text{Hess}_{z_2} \Psi(T, 0)) \neq 0. \end{cases}$$

For our unique critical point $z_1 = 0$ we obtain

$$\begin{cases} \frac{\partial z_2}{\partial t}(t, 0) = -(\partial_{z_2, z_2}^2 \Psi(t, 0, 0))^{-1} \partial_t \partial_{z_2} \Psi(t, 0), \\ \frac{\partial z_2}{\partial z_1}(t, 0) = -(\partial_{z_2, z_2}^2 \Psi(t, 0, 0))^{-1} \partial_{z_1, z_2}^2 \Psi(t, 0). \end{cases}$$

Equations (22) show that $\partial_t z_2(T, 0) = 0$ and $\partial_{z_1} z_2(T, 0) = 0$. Hence near $(T, 0)$ we have $z_2(t, z_1) = \mathcal{O}(\|(t - T, z_1)\|^2)$. We use, again, a decomposition w.r.t. T

$$\Psi(t, z_1, z_2(t, z_1)) = \Psi(T, z_1, z_2(T, z_1)) + (t - T)g(t, z_1, z_2(t, z_1)). \quad (67)$$

Since we have $d_{z_2}g(t, z_1, z_2(t, z_1)) = 0$, it follows that

$$\text{Hess}_{z_1}(g(T, z_1, z_2(T, z_1)))|_{z_1=0} = Q_T.$$

Proposition 23 gives by identification of homogeneous terms of same degree

$$\Psi(T, z_1, z_2(T, z_1)) = R_k(z_1, 0) + \mathcal{O}(\|z_1\|^{k+1}). \quad (68)$$

The stationary phase method in z_2 , see formula (63), shows that

$$I(T, h) \sim \sum_{\nu=0}^N c_\nu h^{n+\nu-d_T} \int A_\nu(t, z_1, 0) e^{\frac{i}{h}(R(t, z_1) + (t-T)g(t, z_1))} dt dz_1, \quad h \rightarrow 0.$$

Copying the construction for a total period we obtain normal forms for the phase $\Psi(t, z_1) = R(t, z_1) + (t - T)g(t, z_1)$, with the decomposition w.r.t. C_{Q_T} :

$$A_0(t, z_1, 0) e^{\frac{i}{h}\Psi(t, z_1)} \simeq \begin{cases} A_{1,i}(\chi_0, \chi_1) e^{\pm \frac{i}{h}\chi_0\chi_1^2} & \text{outside } C_{Q_T}, \\ A_{2,j}(\chi_0, \chi_1, \chi_2) e^{\frac{i}{h}(\chi_0\chi_1^2\chi_2 \pm \chi_1^k)} & \text{near } C_{Q_T} \setminus C_{R_k}, \\ A_{3,l}(\chi_0, \chi_1, \chi_2, \chi_3) e^{\frac{i}{h}(\chi_0\chi_1^2\chi_2 \pm \chi_1^k\chi_3)} & \text{near } C_{Q_T} \cap C_{R_k}. \end{cases}$$

Now the dimension is $l_T = 2d_T$ and results obtained for total periods show again that the contributions of normal forms 2 and 3 are dominating these of normal forms 1. Combining this with the leading term of the stationary phase method gives

$$I(T, h) = (2\pi h)^{n-d_T} \frac{e^{i\frac{\pi}{4}\text{sgn}(q_T)}}{|\det q_T|^{\frac{1}{2}}} \left(\int_{\mathbb{R} \times \mathbb{R}^{l_T}} a(t, z_1, 0) e^{\frac{i}{h}(R(t, z_1) + (t-T)g(t, z_1))} dt dz_1 + \mathcal{O}(h) \right),$$

and the contribution of a non-total period is computed by restriction of all objects to \mathfrak{F}_T . This proves Theorems 3 and 4 in their general forms.

7 Examples

Perturbation of harmonic oscillators. Let be

$$H(x, \xi) = \frac{1}{2}((x_1^2 + \xi_1^2) - (x_2^2 + \xi_2^2)) + (x_2^2 + \xi_2^2)^2, \quad (69)$$

with critical energies 0 and $-\frac{1}{16}$. We consider $\Sigma_0 = \{(x, \xi) / H(x, \xi) = 0\}$, the origin is the only singularity of this surface. The system is integrable with

$$\Phi_t(x, \xi) = \begin{pmatrix} x_1 \cos(t) + \xi_1 \sin(t) \\ x_2 \cos((4(x_2^2 + \xi_2^2) - 1)t) + \xi_2 \sin((4(x_2^2 + \xi_2^2) - 1)t) \\ \xi_1 \cos(t) - x_1 \sin(t) \\ \xi_2 \cos((4(x_2^2 + \xi_2^2) - 1)t) - x_2 \sin((4(x_2^2 + \xi_2^2) - 1)t) \end{pmatrix}.$$

On Σ_0 the function $4(x_2^2 + \xi_2^2) - 1$ is bounded and some elementary arithmetical considerations show that we can find a neighborhood $V(T)$ of the origin, in Σ_0 , such that all periodic trajectory has a period greater than $|T|$. Hence (H_5) is true here. We have

$$d\Phi_t(0) = \begin{pmatrix} \cos(t) & 0 & \sin(t) & 0 \\ 0 & \cos(t) & 0 & -\sin(t) \\ -\sin(t) & 0 & \cos(t) & 0 \\ 0 & \sin(t) & 0 & \cos(t) \end{pmatrix},$$

and 2π is a total period. Expanding the flow in a Taylor-series gives

$$d_0^3 \Phi_{2\pi}((x, \xi)^3) = \begin{pmatrix} 0 \\ 4\xi_2(x_2^2 + \xi_2^2) \\ 0 \\ -4x_2(x_2^2 + \xi_2^2) \end{pmatrix}.$$

Hence $R_4(x, \xi)$ is negative on $C_Q \setminus \{0\}$, since

$$R_4(x, \xi) = \frac{1}{24} \langle (x, \xi), Jd_0^3 \Phi_{2\pi}((x, \xi)^3) \rangle = -\frac{1}{6}(x_2^2 + \xi_2^2)^2.$$

Siegel and Moser example. To illustrate some properties on periodic trajectories, Siegel & Moser introduced in [12] the Hamiltonian

$$H(x, \xi) = \frac{1}{2}(x_1^2 + \xi_1^2) - (x_2^2 + \xi_2^2) + x_1 \xi_1 x_2 + \frac{1}{2}(x_1^2 - \xi_1^2) \xi_2. \quad (70)$$

The origin is the only critical point on $\Sigma_0 = \{(x, \xi) / H(x, \xi) = 0\}$ and trajectories from the surface $\{x_1 = \xi_1 = 0\}$ are 2π -periodic. With $p = (x_1^2 + \xi_1^2)$, $q = (x_2^2 + \xi_2^2)$, we obtain, $p'' = 4pq + p^2$, (see [12]). Since p is strictly positive, p is strictly convex and hence non-periodic. Consequently, on Σ_0 there is no non-trivial periodic trajectories. The linearized flow at the critical point is

$$d\Phi_t(0) = \begin{pmatrix} \cos(t) & 0 & \sin(t) & 0 \\ 0 & \cos(2t) & 0 & \sin(2t) \\ -\sin(t) & 0 & \cos(t) & 0 \\ 0 & -\sin(2t) & 0 & \cos(2t) \end{pmatrix},$$

with a resonance of order 3 since, $2w_1 - w_2 = 0$. For the "period" 2π we have

$$R_3(x, \xi) = \frac{1}{6} \langle -Jd_0^2\Phi_{2\pi}((x, \xi)^2), (x, \xi) \rangle = \frac{\pi}{4} (2x_1x_2\xi_1 + \xi_2(x_1^2 - \xi_1^2)).$$

The intersection $C_Q \cap C_{R_3}$ is obtained by solving the system

$$\begin{cases} (x_1^2 + \xi_1^2) - 2(x_2^2 + \xi_2^2) = 0, \\ 2x_1x_2\xi_1 + \xi_2(x_1^2 - \xi_1^2) = 0. \end{cases}$$

This leads to the surfaces

$$(S_1) : \begin{cases} x_2(x_1, \xi_1) = \frac{x_1^2 - \xi_1^2}{\sqrt{2}x_1\xi_1} \sqrt{\frac{x_1^2\xi_1^2}{x_1^2 + \xi_1^2}} \\ \xi_2(x_1, \xi_1) = -\sqrt{2} \sqrt{\frac{x_1^2\xi_1^2}{x_1^2 + \xi_1^2}} \end{cases}, \quad (S_2) : \begin{cases} x_2(x_1, \xi_1) = -\frac{x_1^2 - \xi_1^2}{\sqrt{2}x_1\xi_1} \sqrt{\frac{x_1^2\xi_1^2}{x_1^2 + \xi_1^2}} \\ \xi_2(x_1, \xi_1) = \sqrt{2} \sqrt{\frac{x_1^2\xi_1^2}{x_1^2 + \xi_1^2}} \end{cases}.$$

By symmetry we just examine gradients on the first surface (S_1) , we have

$$\nabla R_{3|S_1} = \frac{\pi}{4} \begin{pmatrix} -\sqrt{2}x_1\xi_1^2 \sqrt{\frac{x_1^2 + \xi_1^2}{x_1^2\xi_1^2}} \\ 2x_1\xi_1 \\ \sqrt{2}x_1^2\xi_1 \sqrt{\frac{x_1^2 + \xi_1^2}{x_1^2\xi_1^2}} \\ (x_1^2 - \xi_1^2) \end{pmatrix}, \quad \nabla Q|_{S_1} = \begin{pmatrix} 2x_1 \\ 4 \frac{x_1^2 - \xi_1^2}{\sqrt{2}x_1\xi_1} \sqrt{\frac{x_1^2\xi_1^2}{x_1^2 + \xi_1^2}} \\ 2\xi_1 \\ -4\sqrt{2} \sqrt{\frac{x_1^2\xi_1^2}{x_1^2 + \xi_1^2}} \end{pmatrix}.$$

But, since the minor determinant extracted from $\nabla Q|_{S_1}$ and $\nabla R_{3|S_1}$

$$D(x_1, \xi_1) = \left| \begin{pmatrix} x_1 & \xi_1 \\ -x_1\xi_1^2 \sqrt{\frac{x_1^2 + \xi_1^2}{x_1^2\xi_1^2}} & x_1^2\xi_1 \sqrt{\frac{x_1^2 + \xi_1^2}{x_1^2\xi_1^2}} \end{pmatrix} \right| = \sqrt{\frac{x_1^2 + \xi_1^2}{x_1^2\xi_1^2}} x_1\xi_1(x_1^2 + \xi_1^2),$$

is non zero for $(x_1, \xi_1) \neq 0$, $\nabla Q, \nabla R_3$ are linearly independent on $C_Q \cap C_{R_3} \cap \mathbb{S}^3$.

References

- [1] R.Balian and C.Bloch, Solution of the Schrödinger equation in term of classical path, *Annals of Physics* **85** (1974) 514-545.
- [2] R.Brummelhuis, T.Paul and A.Uribe, Spectral estimate near a critical level, *Duke Math. Journal* **78** (1995) no. 3, 477-530.
- [3] R.Brummelhuis and A.Uribe, A semi-classical trace formula for Schrödinger operators, *Commun. Math. Phys.* **136** (1991) no. 3, 567-584.
- [4] A.M.Charbonnel and G.Popov, A semi-classical trace formula for several commuting operators, *Commun. Partial Differential Equations* **24** (1999) no. 1-2, 283-323.
- [5] J.J.Duistermaat, Oscillatory integrals Lagrange immersions and unfolding of singularities, *Commun Pure Appl. Math.* **27** (1974), 207-281.
- [6] V.Guillemin and A.Uribe, Circular symmetry and the trace formula, *Inven. Math.* **96** (1989) no. 2, 385-423.
- [7] M.Gutzwiller, Periodic orbits and classical quantization conditions, *J. Math. Phys.* **12** (1971).
- [8] L.Hörmander, "The analysis of linear partial operators 1,2,3,4", Springer-Verlag (1985).
- [9] L.Hörmander, "Seminar on singularities of solutions of linear partial differential equations", *Annals of mathematical studies* 91, Princeton University Press (1979) 3-49 .
- [10] D.Khuat-Duy, A semi-classical trace formula for Schrödinger operators in the case of a critical energy level, *Thèse de l'université Paris 9*.
- [11] D.Khuat-Duy, A semi-classical trace formula for Schrödinger operators in the case of a critical energy level, *J. Funct. Anal.* **146** (1997) no. 2, 299-351.
- [12] J.K.Moser and C.L.Siegel, "Lectures on celestial mechanics", Springer-Verlag (1995).
- [13] T.Paul and A.Uribe, Sur la formule semi-classique des traces, *Comptes Rendus Séances Acad.Sci. Série I.* **313** (1991) no. 5, 217-222.
- [14] V.Petkov and G.Popov, Semi-classical trace formula and clustering of the eigenvalues for Schrödinger operators, *Ann. Inst. Henri Poincaré Phys. Théorique* **68** (1998) no. 1, 17-83.
- [15] D.Robert, "Autour de l'approximation semi-classique", *Progress in mathematics* Volume 68, Birkhäuser Boston (1987).
- [16] R.Wong, "Asymptotic approximations of integrals", Academic Press Inc. (1989).